Model Solutions

Name:____________________________ **AMS 131 Final**

ID #: ____________________________ **Monday, 9th June, 2014**

**131-01 / 131-02 (circle as appropriate)**

**Read the following instructions carefully.**

- You are advised to real all the questions carefully before beginning to answer any of them.
- This exam is closed book, apart from two 8.5x11 sheets of notes (double sided).
- A table of useful distributions is on the back of this page.
- You must show working or explain all answers for full credit.
- Questions may be worth different numbers of marks. See the table below.
- There are a total of 42 marks available. You can score 100% on this exam by completing questions worth 36 marks.
- It is your choice as to which questions you attempt.
- If you complete questions worth more than 36 marks, you can score more than 100%. You cannot score more than 110%, however.

| | | |
|---|---|---|
| 1. | __________ | (/3) |
| 2. | __________ | (/3) |
| 3. | __________ | (/7) |
| 4. | __________ | (/6) |
| 5. | __________ | (/6) |
| 6. | __________ | (/4) |
| 7. | __________ | (/5) |
| 8. | __________ | (/8) |

| Name | parameters | PMF or PDF | Mean | Variance |
| --- | --- | --- | --- | --- |
| Bernoulli | $p$ | $P(X=1)=p,\ P(X=0)=1-p$ | $p$ | $p(1-p)$ |
| Binomial | $n,p$ | $\binom{n}{k}p^k(1-p)^{n-k}$ for $k=0,1,\dots,n$ | $np$ | $np(1-p)$ |
| Hypergeometric | $w,b,n$ | $\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$, for $k=0,1,\dots,n$ | $\frac{nw}{w+b}$ | $\frac{nwb}{(w+b)^2}\frac{w+b-n}{w+b-1}$ |
| Geometric | $p$ | $p(1-p)^k$, for $k=1,2,\dots$ | $\frac{1-p}{p}$ | $\frac{1-p}{p^2}$ |
| Negative Binomial | $r,p$ | $\binom{r+n-1}{r-1}p^r(1-p)^n$, for $n=0,1,\dots$ | $\frac{r(1-p)}{p}$ | $\frac{r(1-p)}{p^2}$ |
| Poisson | $\lambda$ | $\frac{e^{-\lambda}\lambda^k}{k!}$, for $k=0,1,\dots$ | $\lambda$ | $\lambda$ |
| Uniform | $a,b$ | $\frac{1}{b-a}$, for $a\le x\le b$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |
| Beta | $a,b$ | $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$, for $0<x<1$ | $\frac{a}{a+b}$ | $\frac{ab}{(a+b)^2(a+b+1)}$ |
| Normal | $\mu,\sigma^2$ | $\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ | $\mu$ | $\sigma^2$ |

3

1. Show that $E[\cos^4(X^2+1)] \geq (E[\cos^2(X^2+1)])^2$.

Let $Y = \cos^2(x^2+1)$.

$Var(Y) = E(Y^2) - (E(Y))^2 \geq 0$

Hence $E(Y^2) \geq (E(Y))^2$.

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2. On your iPod you have 10 songs, five from each of two albums. Whilst working on the review problems you hit the shuffle button and listened to 6 of the songs. What is the probability that three of the songs that you listened to came from each album?

Hypergeometric probability

$$\frac{\binom{5}{3}\binom{5}{3}}{\binom{10}{6}}. \qquad \frac{\left(\text{3 out of 5 from 1 album}\right)\left(\text{3 out of 5 from other album}\right)}{\left(\text{6 out of 10 songs}\right)}$$

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3. Your friend has a set of fair dice, with 6 of them having 6 sides, 5 of them having 8 sides, and 4 of them having 12 sides. The sides of each die are numbered 1 through the number of sides on the die. Your friend choses one from their set at random, rolls it 11 times, and gets one six.

   (a) What is the most likely number of sides on the die they have rolled?

   (b) Would your answer change if they told you that they rolled the die 10 times before the first 6 appeared?

1 a) Need to find P(die has 6 sides | one 6 out of 11 rolls)

P( 8 | )

P( 12 | ).

1 $P(\text{die has } k \text{ sides} \mid \text{one 6 out of 11 rolls})$

$$= \frac{P(\text{one 6 out of 11 rolls} \mid \text{die has } k \text{ sides})\, P(\text{die has } k \text{ sides})}{\sum_{k=6,8,12} P(\text{one 6 out of 11 rolls} \mid \text{die has } k \text{ sides})\, P(\text{die has } k \text{ sides})}$$

1

$P(6 \mid \text{data}) \propto \binom{11}{1} \frac{1}{6} \times \left(\frac{5}{6}\right)^{11} \times \frac{6}{15} \propto \frac{1}{15}\left(\frac{5}{6}\right)^{11} = 0.0089725$

$P(8 \mid \text{data}) \propto \binom{11}{1} \frac{1}{8} \times \left(\frac{7}{8}\right)^{11} \times \frac{5}{15} \propto \frac{5}{15} \times \frac{1}{8} \times \left(\frac{7}{8}\right)^{11} = 0.00959..$

$P(12 \mid \text{data}) \propto \binom{11}{1} \frac{1}{12} \times \left(\frac{11}{12}\right)^{11} \times \frac{4}{15} \propto \frac{1}{3} \times \frac{1}{12} \times \left(\frac{11}{12}\right)^{11} = 0.0085332$

1 $P(\text{six sided} \mid) = \frac{0.0089}{0.0089 + 0.00959 + 0.0085} = ~~0.3304~~\ 0.331$

$P(8 \text{ sided} \mid) = ~~0.3548~~\ 0.354$

$P(12 \text{ sided} \mid) = 0.315$

1 ⇒ Most likely to be 8 sided.

b) No - form of the likelihood is identical, only the constant has changed

2

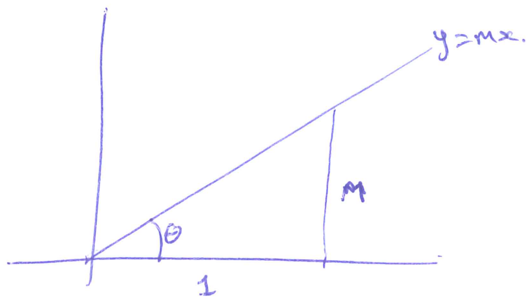
$\boxed{6}$

4. A straight line through the origin makes an angle $\theta$ with the x-axis, where $\theta$ is uniformly distributed between 0 and $\pi/2$.

   (a) What is the corresponding pdf for the slope of the line $M$? (The line can be expressed as $Y = Mx$.) Together with the functional form of the pdf, give the range of values for which the pdf is non-zero.

   $y = ax$

   (b) What is the expected value of $M$?




(1) $\tan\theta = m$ $\quad\quad \theta = \arctan(m)$.

$$\frac{d\theta}{dM} = \frac{1}{1+M^2}$$

(1) $f_M(M)\,dM = f_\theta(\theta)\,d\theta$

$$f_M(M) = f_\theta(\theta)\frac{d\theta}{dM} \quad\quad f(\theta) = \frac{2}{\pi} \quad 0 < \theta < \frac{\pi}{2}$$

(1) $f_M(M) = \frac{2}{\pi}\frac{1}{1+M^2}$ $\quad\quad$ (1) $0 < M < \infty$

b) (1) $E(M) = \int_0^\infty M \frac{2}{\pi} \frac{1}{1+M^2}\, dM$

$$= \frac{2}{\pi}\int_0^\infty \frac{M}{1+M^2}\, dM \quad\quad \text{let } u = 1+M^2$$

$du = 2M\,dM$.

$M = 0, \quad u = 1$

$M = \infty \quad u = \infty$

$$= \frac{1}{\pi}\int_1^\infty \frac{1}{u}\, du$$

(1) The expectation does not exist.

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5. Random variable $X$ has pdf $f_X(x) = 3\exp(-3x)$, for $x > 0$. Let $Y = e^X$.

(a) Find the mean and variance of $Y$.

(b) For $Y_1, ..., Y_n$ i.i.d. with the same distribution as $Y$, what is the approximate distribution of the sample mean $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ when $n$ is large?

(2) a) Mean $= E(Y) = \int_0^\infty e^x \cdot 3 e^{-3x}\, dx$ LOTUS.

$= 3\int_0^\infty e^{-2x}\, dx$

$= -\frac{3}{2} e^{-2x}\Big|_0^\infty = -\frac{3}{2}(0-1) = \frac{3}{2}.$

(1) $E(Y^2) = \int_0^\infty (e^x)^2 \cdot 3e^{-3x}\, dx = \int_0^\infty 3e^{-x}\, dx.$

$= -3 e^{-x}\Big|_0^\infty = 3.$

(1) $\text{var}(Y) = E(Y^2) - (E(Y))^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{12-9}{4} = \frac{3}{4}.$

b) By CLT $\bar{Y}_n$ is Normal, mean = mean of Y

var = $\frac{\text{var of } Y}{n}$

(2) $\bar{Y}_n \sim N\left(\frac{3}{2}, \frac{3}{4n}\right).$

4

6. a fair die is rolled twice, with outcomes $X$ for the first roll and $Y$ for the second roll. Let $W = X + Y$ and $Z = X - Y$.

   (a) Compute the covariance of $W$ and $Z$.

   (b) Are $W$ and $Z$ independent? Explain your answer clearly.

a) $\mathrm{Cov}(W, Z) = \mathrm{Cov}(X, +Y, X-Y)$

2

$= \mathrm{Cov}(X,X) + \mathrm{Cov}(Y,X) - \mathrm{Cov}(X,Y) - \mathrm{Cov}(Y,Y).$

$= 0.$

b) No. knowing the sum can tell you something about the difference

2

eg if $X + Y = 2$.

then $x = 1$, $Y = 1$ and $Z = X - Y = 0$.

5

7. For a certain type of person, the sea in Santa Cruz has two states, "surf's up", and "surf's down". If surf is up on a particular day, it is up the next day with probability 0.7. If surf is down on a particular day, it is down on the next day with probability 0.4.

   (a) If surf's up today (Monday), what's the probability that surf's up on Wednesday this week?

   (b) What proportion of all days are "surf up" days?

$$\begin{array}{ccc} & & \text{to} \\ & & \begin{array}{cc} \text{up} & \text{down} \end{array} \\ \text{from} & \begin{array}{c} \text{up} \\ \text{down} \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

1 a) need to compute $\underline{s} T^2$ where $\underline{s} = [1 \quad 0]$

$$\underline{s}T = [1 \quad 0]\begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = [0.7 \quad 0.3]$$

1 $$\underline{s}T^2 = [0.7 \quad 0.3]\begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = [0.67 \quad 0.33].$$

$$\begin{array}{r} 0.49 + \\ 0.18 \\ \hline .67 \end{array} \qquad \begin{array}{r} 0.21 \\ 0.12 \\ \hline .33 \end{array}$$

1 $\Rightarrow$ surf's up on wed with prob. 0.67.

b) stationary distribution

$(T - I)$ + replace last column by 1's.

1 $$G = \begin{bmatrix} -0.3 & 1 \\ 0.6 & 1 \end{bmatrix} \qquad G^{-1} = \frac{1}{-0.3 - 0.6}\begin{bmatrix} 1 & -1 \\ -0.6 & -0.3 \end{bmatrix}.$$

+ stat dist is bottom row, ie $\frac{1}{0.9}[0.6 \quad 0.3]$.

1 ie $\left[\frac{2}{3} \quad \frac{1}{3}\right]$

$\Rightarrow$ surf's up $\frac{2}{3}$ of the time.

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8. On the midterm, we showed that the expected number of packets of lego mini figures that my son needs to buy to collect the whole set was approximately $n \log_e(n)$, where $n$ is the number of distinct mini figures.

   (a) Let $X$ be the random variable that is the number of packets bought before obtaining a full set. By again considering $X$ as the sum of simpler random variables, show that the variance of the number of packets needed to obtain a full set is

$$n \sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$$

.

   (b) There are 8 mini figures in the series that my son is trying to collect. What is an approximate value for the probability that he will have to buy more than 34 packets to get a full set?

a) 1 $X = X_1 + X_2 + \dots + X_n$

~~$X_1$~~ # packets to 1st minifigure

$X_2$ # of additional packets to 2nd minifigure

⋮

$X_n$ # of additional packets to $n$th minifigure.

$X_i$ is 1st success dist with ~~prob~~. success prob. $\frac{n-(i-1)}{n}$

1 $X_i$'s are independent.

1 $\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) + \dots \text{Var}(X_n)$.

1 Variance of 1st success is same as variance of Geometric

$\frac{1-p}{p^2}$

$\text{Var}(X) = \sum_{i=1}^{n} \cancel{\frac{1 - \frac{n-(i-1)}{n}}{\left(\frac{n-(i-1)}{n}\right)^2}} = \cancel{\frac{n^2}{(n-(i-1))^2} - \frac{n}{n(i-1)}}$

$= \cancel{\frac{n^2 - n^2(i-1)}{(n-(i-1))^2}} = n^2$

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$$\text{Var}(X) = 0 + \frac{1-\frac{n-1}{n}}{\left(\frac{n-1}{n}\right)^2} + \frac{1-\frac{n-2}{n}}{\left(\frac{n-2}{n}\right)^2} + \dots + \frac{1-\frac{1}{n}}{\left(\frac{1}{n}\right)^2}$$

$$= \frac{n}{(n-1)^2} + \frac{2n}{(n-2)^2} + \frac{3n}{(n-3)^2} + \dots + \frac{(n-1)n}{1^2}$$

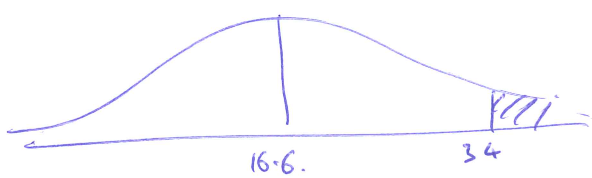
$$= n\left[\frac{1}{(n-1)^2} + \frac{2}{(n-2)^2} + \frac{3}{(n-3)^2} + \dots \frac{n-1}{1}\right]$$

1 $$= n\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$$

b) 1 For $n = 8$ mean is $8 \times \log_e 8 = 16.6$.

1 var $= 76.7$ (from above). (76.0 ?).

$\Rightarrow$ std $= 8.76$. (8.71 ?)

1 


1

34 is mean + 2sd.

$\Rightarrow$ prob (buy more than 34) is shaded area

$= 0.025$

(or $1 - \Phi(2)$).