Name:____________________________ **AMS 131 Final**

ID #: ____________________________ **Wednesday, 13th June, 2018**

AMS 131

Read the following instructions carefully.

- You are advised to real all the questions carefully before beginning to answer any of them.
- This exam is closed book, apart from two 8.5x11 sheets of notes (double sided).
- A table of useful distributions is on the back of this page.
- You must show working or explain all answers for full credit.
- Questions may be worth different numbers of points. See the table below.
- There are a total of 42 points available. You can score 100% on this exam by completing questions worth 34 pointss.
- It is your choice as to which questions you attempt.
- If you complete questions worth more than 34 points, you can score more than 100%. You cannot score more than 110%, however.

1.	__________	(/8)
2.	__________	(/3)
3.	__________	(/4)
4.	__________	(/3)
5.	__________	(/9)
6.	__________	(/7)
7.	__________	(/4)
8.	__________	(/4)

Name	parameters	PMF or PDF	Mean	Variance
Bernoulli	p	$P(X=1)=p,\ P(X=0)=1-p$	p	$p(1-p)$
Binomial	n, p	$\binom{n}{k}p^k(1-p)^{n-k}$ for $k=0,1,\dots,n$	np	$np(1-p)$
Hypergeometric	w, b, n	$\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$, for $k=0,1,\dots,n$	$\frac{nw}{w+b}$	$\frac{nwb}{(w+b)^2}\frac{w+b-n}{w+b-1}$
Geometric	p	$p(1-p)^k$, for $k=1,2,\dots$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
Negative Binomial	r, p	$\binom{r+n-1}{r-1}p^r(1-p)^n$, for $n=0,1,\dots$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$
Poisson	λ	$\frac{e^{-\lambda}\lambda^k}{k!}$, for $k=0,1,\dots$	λ	λ
Uniform	a, b	$\frac{1}{b-a}$, for $a \le x \le b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Beta	a, b	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$, for $0<x<1$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$
Normal	μ, σ^2	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$	μ	σ^2

~~5~~ 8

1. There are 100 shoelaces in a box. You randomly pick two ends (which may or may not belong to the same shoelace) and tie them together. You either end up with a loop (if the two ends belonged to the same shoelace), or a longer shoelace (if they didn't). You repeat this process until only loops remain.

   (a) How many times must the process be repeated until only loops remain? (Think about the reduction in the number of non-loop shoelaces at each step.)

   (b) Show that the expected number of loops when only loops remain is given by

$$\sum_{n=1}^{100} \frac{1}{2n-1}$$

   (For each step, create an indicator random variable for whether a loop was created; think about the probability of creating a loop; and think about the reduction in the number of free ends.)

1 a) 100 steps. At each step the number of non-loop shoelaces is reduced by 1 by either

- forming a loop
- joining two laces into a longer lace.

b) Let X_n be an indicator RV for the event

1 "a loop is formed at the n^{th} step".

1 then $X = \sum_{i=1}^{100} X_n$ and $E[X]$ = expected number of loops when only loops remain

$$= \sum_{i=1}^{100} E[X_n]$$

1

1 X_n is an indicator RV, so $E[X_n]$ is the probability of the event which X_n is an indicator for.

PTO

[Blank Page]

(1) $E[X_n]$ = prob. of forming a loop ~~from~~ by randomly chosing two ends from a pile of n shoelaces

$$= \frac{n}{\binom{2n}{2}}$$

— n ways to pick both ends of ~~the~~ one of the n shoelaces

— # ways of chosing 2 ends from $2n$ ends

$$\Rightarrow \binom{2n}{2} = \frac{(2n)!}{2!\,(2n-2)!} = \frac{2n(2n-1)}{2}$$

(1) $$\Rightarrow E[X_n] = \frac{n}{n(2n-1)} = \frac{1}{2n-1}$$

(1) $$\Rightarrow E[X] = \sum_{n=1}^{100} \frac{1}{2n-1}$$

3

2. In my ~~youngest~~ son's preschool, there are 2 classes in his year, 1A and 1B. Each class has 5 kids. For next year, one of the classes (2A, say) will have 6 kids. If the 6 kids are chosen randomly from the 10 kids in year 1, what is the probability that class 2A will have 3 kids from each of class 1A and 1B?

Hypergeometric.

$$\frac{\binom{5}{3} \times \binom{5}{3}}{\binom{10}{6}}$$

$$\frac{\left(\begin{array}{c}\text{3 out of}\\ \text{5}\\ \text{from}\\ \text{one class}\end{array}\right)\left(\begin{array}{c}\text{3 out of}\\ \text{5 from}\\ \text{other class}\end{array}\right)}{\left(\begin{array}{c}\text{6 out of}\\ \text{10 kids}\end{array}\right)}.$$

4

independently on each toss.

3. I have a biased coin that has probability p of coming up heads. I toss it N times, and it comes up heads k times out of the N. What is the probability that the first toss came up heads?

Let $X_1 \dots X_N$ be the N tosses. - ~~Ber~~ each iid Bernoulli(p)

Then we want

(1) $$P\left(X_1 = 1 \,\middle|\, \sum_{i=1}^{N} X_i = k\right)$$

(1) $$= \frac{P\left(X_1 = 1 \text{ and } \sum_{i=1}^{N} X_i = k\right)}{P\left(\sum_{i=1}^{N} X_i = k\right)}$$

(1) $$= \frac{P(X_1 = 1)\, P\left(\sum_{i=2}^{N} X_i = k-1\right)}{P\left(\sum_{i=1}^{N} X_i = k\right)}$$

$$= \frac{P(\text{H on 1st toss})\,(\text{k-1 heads out of N-1 tosses})}{P(\text{k heads out of N tosses})}$$

(1) $$= \frac{p \times \binom{N-1}{k-1} p^{k-1} (1-p)^{N-1-(k-1)}}{\binom{N}{k} p^k (1-p)^{N-k}} = \frac{\binom{N-1}{k-1}}{\binom{N}{k}} = \frac{k}{N}$$

$\boxed{3}$

4. On a certain road, three out of every four trucks are followed by a car, while only one out of every five cars is followed by a truck. What proportion of the vehicles on the road are trucks?

Transition matrix

(1)

$$\begin{array}{ccc} & & \text{To} \\ & \text{Car} & \text{Truck} \\ \text{from} \quad \text{Car} & & \\ \text{Truck} & & \end{array} \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ 3/4 & \frac{1}{4} \end{bmatrix}$$

stationary distribution

(1)

$$\begin{pmatrix} a & 1-a \end{pmatrix} \begin{bmatrix} 4/5 & 1/5 \\ 3/4 & 1/4 \end{bmatrix} = \begin{pmatrix} a & 1-a \end{pmatrix}$$

$$\frac{4}{5}a + \frac{3}{4}(1-a) = a$$

$$\frac{3}{4} = a + \frac{3}{4}a - \frac{4}{5}a$$

$$\frac{3}{4} = \frac{19}{20}a$$

$$a = \frac{15}{19}.$$

$\Rightarrow$ Proportion of vehicles that are <u>cars</u> is $\frac{15}{19}$

(1) $\Rightarrow$ proportion that are <u>trucks</u> $= 1 - \frac{15}{19} = \underline{\underline{\frac{4}{19}}}.$

9

5. In one of the homework problems, we had that Alice is trying to transmit to Bob the answer to a yes-no question, using a noisy channel. She encodes "yes" as 1 and "no" as 0, and sends the appropriate value. However, the channel adds noise; specifically, Bob receives what Alice sends plus a $N(0, \sigma^2)$ noise term (the noise is independent of what Alice sends).

In the homework problem, it was equally likely for Alice to transmit a 1 or a 0; in this case it is optimal (in terms of Bob maximizing the probability of correctly understanding Alice) for Bob to use a threshold of 1/2; that is, if Bob receives a value greater than 1/2 he interprets it as yes; otherwise, he interprets it as no.

Consider now the case where Alice sends a 1 with probability p, and a zero with probability $1-p$. Assume that Bob knows the value of p.

Bob must now decide on the threshold that maximizes the probability that he will understand Alice correctly.

For a threhsold of t, if Bob recieves a value greater than t he interpets it as a "yes"; otherwise he interprets it as a "no".

(a) Show that the probability that Bob correctly understands Alice is

$$p\left(1-\Phi\left(\frac{t-1}{\sigma}\right)\right)+(1-p)\Phi\left(\frac{t}{\sigma}\right)$$

where $\Phi()$ is the CDF of the standard normal distribution.

(b) Show that the optimal threshold is given by the solution to an equation of the form

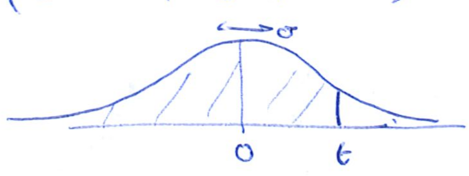
$$(1-p)f\left(\frac{t}{\sigma}\right)=pf\left(\frac{t-1}{\sigma}\right)$$

where you must specify $f()$.

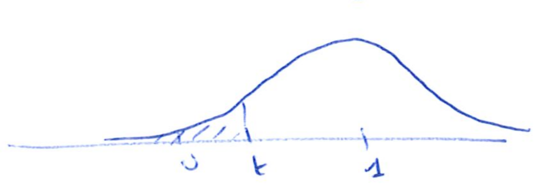
(c) Based on the equation in 5b, draw a sktech that indicates the optimal threshold, and explain how it is compatible with the result that the optimal threshold is 1/2 if $p = 1/2$.

(1) a) $P(\text{correct}) = P(\text{correct} \mid 1 \text{ is sent}) \times P(1 \text{ is sent})$
$+\, P(\text{correct} \mid 0 \text{ is sent}) \times P(0 \text{ is sent})$

(1) $P(\text{correct} \mid 0 \text{ is sent}) = \Phi(t/\sigma)$.



(1) $P(\text{correct} \mid 1 \text{ is sent}) = 1 - \Phi\left(\frac{t-1}{\sigma}\right)$.



[Blank Page]

$$\Rightarrow \quad p(\text{correct}) = p\left(1 - \Phi\left(\frac{t-1}{\sigma}\right)\right) + (1-p)\,\Phi\left(\frac{t}{\sigma}\right)$$

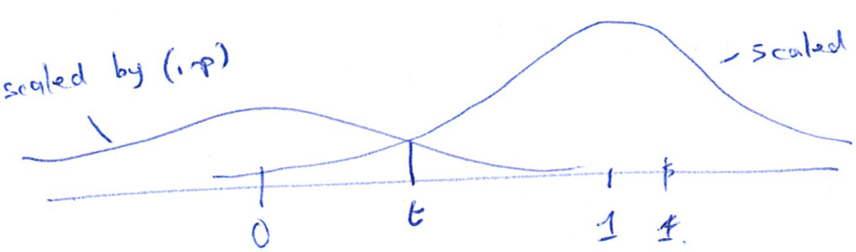
b) The optimal threshold is the one that maximises $p(\text{correct})$ as a fn of t.

(1) $$\frac{d}{dt}\,p(\text{correct}) = -\frac{p}{\sigma}\,f_T\left(\frac{t-1}{\sigma}\right) + \frac{(1-p)}{\sigma}\,f_T\left(\frac{t}{\sigma}\right) = 0 \quad \text{for max/min}$$

(1) Hence $(1-p)\,f_t\left(\frac{t}{\sigma}\right) = p\,f_T\left(\frac{t-1}{\sigma}\right)$

(1) where $f_t()$ is the standard normal pdf.

c)

(1) 

both are Normal with same σ.

(1) optimal threshold is where they cross.

(1) if $p = \frac{1}{2}$, each curve is scaled the same, and so they cross at the mid point between 0 and 1, ie optimal threshold is $\frac{1}{2}$ – as before

7

~~8~~

6. Consider n i.i.d. random variables $X_1, X_2, \ldots, X_n$. Let $X_{(1)} = \min(X_1, \ldots, X_n)$.

(a) By considering the event $X_{(1)} > x$, show that the CDF of $X_{(1)}$ is

$$F_{X_{(1)}}(x) = 1 - (1 - F(x))^n$$

where $F(x)$ is the CDF of each of the X_i's.

(b) If $X_i \sim U(0,1)$, show that the pdf of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = n(1-x)^{n-1}$$

.

(c) This is a degenerate form of which distribution?

(1) a) $\text{CDF} = P(X_{(1)} \leq x) = 1 - P(X_{(1)} > x)$

$$P(X_{(1)} > x) = P(X_1 > x, X_2 > x, \ldots X_n > x)$$

$$= P(X_1 > x)\, P(X_2 > x) \ldots P(X_n > x) \quad \text{(iid)}.$$

(1) $$= (1 - P(X_1 \leq x))(1 - P(X_2 \leq x)) \ldots (1 - P(X_n \leq x))$$

$$= (1 - F(x))^n$$

(1) Hence $\text{CDF} = 1 - (1 - F(x))^n$

(1) b) $$f_{X_{(1)}}(x) = \frac{d}{dx} F_{X_{(1)}}(x) = \frac{d}{dx}\left(1 - (1 - F(x))^n\right).$$

$$= -n (1 - F(x))^{n-1} \times -f_X(x)$$

$$= n f_X(x) (1 - F(x))^{n-1}$$

(1) for $f_X(x) \sim U(0,1)$ ie $f_X(x) = 1$

$F(x) = x$

(1) $\Rightarrow f_{X_{(1)}}(x) = n(1-x)^{n-1}$

(1) c) this is a degenerate Beta distribution $(a=1)$.

$\boxed{\cancel{8}}$ $\boxed{4}$

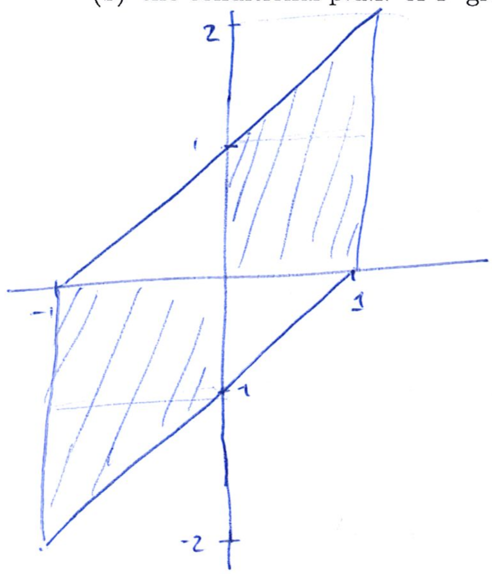
7. Suppose that the joint distribution of X and Y is uniform over the region in the xy-plane bounded by the four lines $x = -1$, $x = 1$, $y = x + 1$ and $y = x - 1$. Determine

   (a) $Pr(XY > 0)$

   (b) the conditional p.d.f. of Y given that $X = x$.

c) are X and Y independent?

① a)



pdf is uniform over this region.

$XY > 0$ is the shaded region

The shaded area is 3

the unshaded area is ~~4~~ 1

$\Rightarrow$ pdf is $\frac{1}{4}$ in the shaded area

$\Rightarrow P(XY > 0) = \frac{3}{4}$

①

① b) Knowing x, Y is restricted to the vertical line between the limits defined by the x-value

ie uniform over the range $[x-1, x+1]$

$$\Rightarrow \quad f_{Y|x}(y|x) = \begin{cases} 1/2 & x-1 < y < x+1 \\ 0 & \text{otherwise.} \end{cases}$$

① c) they are <u>not</u> independent, as the conditional distribution $f_{Y|x}(y|x)$ depends on x.

14

8. Suppose that a pair of balanced dice are rolled 120 times, and let X denote the number of rolls on which the sum of the two numbers is 7. Determine a value of k such that $Pr(|X - 20| \leq k)$ is approximately 0.95.

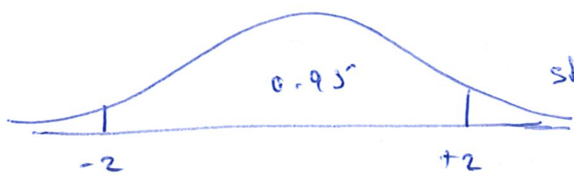
$P(7 \text{ on two dice}) = 1/6$

$\Rightarrow$ By Central limit theorem

(1) $X \sim$ Normal with mean $120 \times \frac{1}{6} = 20$

variance $120 \times \frac{1}{6} \times \frac{5}{6}$

(1) std. dev ≈ 4.08.



standard normal

(1) with prob 0.95 a standard normal RV is within ~~± 2~~ ~ 2 std. dev of the mean.

(1) $\Rightarrow k \approx 2 \times 4.08 = \underline{8.16}$. (or 8 if 1.96 std dev)