Name:____________________ Section: (day/time) ____________________

AMS131-01 - MIDTERM
Thursday 8th May, 2014.

- You must explain all answers and/or show working for full credit.
- This exam is closed book, but you may use one 8.5 by 11 piece of paper with notes, and a calculator.
- This exam is to be completed individually.

1. [9 points] Prove that

(a)

$$n(n-1)\binom{n-2}{k-2} = k(k-1)\binom{n}{k}$$

Consider the number of ways of chosing a comittee of k people from a group of n, with one comittee member designated as chair, and one as secretary. 1

The LHS is the # ways of chosing the chair, then the secretary, then the remaining k-2 ctte members from the remaining n-2 people. 1

the RHS - first chose the k members, then chose 1 to be chair and one to be secretary. 1

(b)

$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$$

consider chosing h+k people out of n. 1

LHS - first chose h, then chose k from the remaining n-h. 1

RHS - first chose k, then chose ~~k~~ h from remaining n-k. 1

(c)

$$\sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

(1) Using the result ~~...~~ $\sum_{k=0}^{j} \binom{m}{k}\binom{n}{j-k} = \binom{m+n}{j}$ ← we proved this in class.

(1) let $j = n$, $m = n$ $\qquad \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$.

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \binom{2n}{n}.$$

(1) $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

2. [4 points] So far we have covered 10 topics in class. I chose 6 of them to be on the exam. You revised 5 topics. You'll get a decent score if 3 of the topics you revised come up on the exam. What's the probability that exactly 3 of the topics you revised will come up on the exam?

This is the same as the question on the quiz

topics → cards.

revised topics → red cards

un-revised topics → blue cards.

I chose 6 → deal 6 cards.

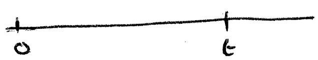
P(exactly 3 of the topics you revised coming up) $= \dfrac{\binom{5}{3} \times \binom{5}{3}}{\binom{10}{6}}$ $\qquad \dfrac{\binom{\text{3 out of 5}}{\text{revised topics}}\binom{\text{3 out of 5}}{\text{un-revised topics}}}{\binom{\text{6 out of 10}}{\text{topics}}}$

(1) - answer

(1) + (1) + (1) for explaining each term.

3. [13 points] Emails arrive at a rate λ per hour. Let T be the random variable that is the time of arrival of the first email.

   (a) By modeling the email arrivals as a Poisson process, and by considering $P(T \geq t)$, derive the CDF of T, and hence show that the pdf of the waiting time is $f_T(t) = \lambda e^{-\lambda t}, t > 0$.



T - time of arrival of 1st email $\Rightarrow$ zero emails in period $0 \to t$.

Poisson process $P(K=k) = \dfrac{e^{-\lambda t}(\lambda t)^k}{k!}$ 1.

Hence $P(T > t) = P(K=0) = \dfrac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$. 1

CDF is $P(T \leq t) = 1 - P(T > t) = 1 - e^{-\lambda t}$ 1

pdf is $\dfrac{d}{dt}(1 - e^{-\lambda t}) = \lambda e^{-\lambda t}$. 1

   (b) Show that the expected waiting time is $1/\lambda$.

$$E[T] = \int_0^\infty t \times \lambda e^{-\lambda t}\, dt.$$ 1

$$= \lambda \int_0^\infty \underbrace{t}_{u}\, \underbrace{e^{-\lambda t}\, dt}_{dv}$$

$$= \lambda \left(\frac{t e^{-\lambda t}}{-\lambda} \Big|_0^\infty - \int_0^\infty \frac{e^{-\lambda t}}{-\lambda}\, dt \right.$$

$$= \lambda \left(0 + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t}\, dt \right).$$

$$= \frac{e^{-\lambda t}}{-\lambda} \Big|_0^\infty = \frac{1}{\lambda}.$$ 2

(c) Show that the variance of the waiting time is $1/\lambda^2$.

1 $\quad \mathrm{Var}(T) = \mathcal{E}(T^2) - (\mathcal{E}(T))^2 \qquad \mathcal{E}(T) = \frac{1}{\lambda}$ hence need $\mathcal{E}(T^2)$.

$$\mathcal{E}(T^2) = \int_0^\infty t^2 \lambda e^{-\lambda t}\, dt$$

$$= \lambda \left. \frac{t^2 e^{-\lambda t}}{-\lambda} \right|_0^\infty - \lambda \int_0^\infty \frac{2t\, e^{-\lambda t}}{-\lambda}\, dt.$$

1 $$= 0 + 2\int_0^\infty t e^{-\lambda t} = \frac{2\,\mathcal{E}(T)}{\lambda} = \frac{2}{\lambda^2}$$

1 $\quad$ Hence $\mathrm{Var}(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

(d) You are waiting for an email, and have already waited time t_0. What is the pdf of the *additional* waiting time until the next email arrives? [Hint: find $P(T \geq t + h | T \geq t)$]

1 $$P(T \geq t+h \mid \cancel{\phi}\, T \geq t) = \frac{P(T \geq t+h \text{ and } T \geq t)}{P(T \geq t)}.$$

1 $$= \frac{P(T \geq t+h)}{P(T \geq t)}.$$

In part a) we showed that $P(T > x) = \cancel{*}\ e^{-\lambda t}$

Hence

1 $$P(T \geq t+h \mid T \geq t) = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = \underline{\underline{e^{-\lambda h}}}.$$

4. [9 points] To go with the Lego Movie, Lego sell minifgures of the characters from the movie. They are sold in packets, where each packet contains one minifigure, and from the outside of the packet it is impossible to tell which minifigure is inside. There are n minifigures to collect.

(a) Assuming that each packet that my son buys is equally likely to contain any one of the minifigures, show that the expected number of packets that he needs to buy to collect the whole set is approximately $n \log n$.

[Hint: express the random variable that is the total number of packets as a sum of simpler terms, and use linearity of expectation.]

X - total # packets

$X = X_1 + X_2 + \dots + X_n$

X_1 # packets until get 1st minifigure

X_2 # packets after 1st until get 2nd (ie new) minifigure (1)

X_3 2nd 3rd

⋮

$X_1 = 1$

$X_2 \sim$ first success with $p = \frac{n-1}{n}$

$X_3 \sim$ first success with $p = \frac{n-2}{n}$ (1)

⋮

$X_n \sim$ first success with $p = \frac{1}{n}$

$E[X] = E[X_1 + X_2 + \dots + X_n]$

$= E[X_1] + E[X_2] + \dots E[X_n]$ (1)

$= 1 + \frac{n}{n-1} + \frac{n}{n-2} + \dots + \frac{n}{1}$ (1)

$= n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2}\ 5 + \dots + 1\right)$

$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) \doteq n \log n.$ (1)

(b) If, in addition to the 16 characters from the movie, there is also a special, gold, character, which is present in one in six hundred of the packets, give an approximate value for the number of packets required to collect the full set together with the gold character. Explain your answer clearly.

(1) expected # packets to get gold figure $= 600$
(from first success distribution with $p = \frac{1}{600}$).

(1) Expected # to collect the other 16 is $16 \log 16$ ~~≤~~ $<< 600$

(1) So, in buying the 600 ~~to get~~
you expect to already have a complete set.

(1) ⇒ Expected # of packets to get whole
set including gold minifigure $\approx \underline{\underline{600}}$.