Review Notes.

3 definitions frequency
classical - equally likely outcomes
Subjective

Experiments + events

set S - all outcomes of the experiment

events are subsets.

1) S is an event.

2) A is an event $\Rightarrow A^c$ is an event

3) Countable union of events is an event.

Union / intersection / complement

$A \cup B$ $\quad A \cap B \quad A^c$

$A \cap B = \emptyset$ - disjoint

De Morgan $\quad (A_1 \cup A_2)^c = A_1^c \cap A_2^c$

$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$

classical prob. - counting

$$P(A) = \frac{\#\text{ favourable outcomes}}{\#\text{ outcomes}}$$

multiplication rule for # outcomes

$\binom{n}{k}$ # ways of chosing k items from n where order does not matter.

Full house $\quad \dfrac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}}$ $\quad$ Cards ~~Balls~~ example on quiz 1.

identities $\quad \binom{n}{k} = \binom{n}{n-k}$

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

chose k out of n.

		order matters	doesn't
replacement	with	n^k	$\binom{n+k-1}{k}$
	without	$n(n-1)\dots(n-k+1)$	$\binom{n}{k}$

k indistinguishable balls in n distinguishable boxes

k balls

$n-1$ separators.

hence $\binom{n+k-1}{k}$

Birthday Problem

P(no two share a bday)

$$= \frac{365}{365} \times \frac{364}{365} \times \dots \frac{365-n+1}{365}$$

$P(\text{match}) = 1 - P(\text{no match})$

Non-nieve Definition.

S - sample space.

P - fn that takes event $A \subseteq S$

and returns $P(A) \in [0,1]$

$P(\emptyset) = 0$

$P(S) = 1$

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ if A_i disjoint

$P(A^c) = 1 - P(A)$.

$A \subseteq B$ $\quad P(A) \leq P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ if A, B not disjoint.

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$
$\quad - P(A \cap B) - P(A \cap C) - P(B \cap C)$
$\quad + P(A \cap B \cap C)$

$P(A_1 \cup A_2 \cup A_3 \dots \cup A_n)$ – by inclusion / exclusion

$\vdots$

$$(-1)^{n+1} \, P(A_1 \cap A_2 \cap A_3 \dots \cap A_n)$$

Matching problem

P(at least one person gets right letter) $\simeq 1 - \frac{1}{e}$

Independence.

$P(A \cap B) = P(A)\,P(B)$ – not the same as disjoint

A, B, C – all pairs, triple must be independent

Conditional Prob.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

all probs are conditional

eg

$P(k \text{ successes}) = \binom{n}{k} p^k q^{n-k}$

is really $P(k \text{ successes} \mid n, p)$

$$P(A \cap B) = P(A \mid B)\,P(B) = P(B \mid A)\,P(A)$$

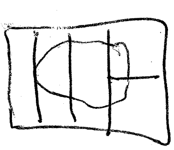
$\Rightarrow$ Bayes Thm $\quad P(B \mid A) = \dfrac{P(A \mid B)\,P(B)}{P(A)}$

$$P(A_1, A_2 \dots A_n) = P(A_n \mid A_1 \dots A_{n-1})\,P(A_{n-1} \mid A_1 \dots A_{n-2}) \dots P(A_3 \mid A_1 A_2)\,P(A_2 \mid A_1)\,P(A_1)$$

$P(A)$ – LOTP.

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \dots + P(A \cap B_n)$$
$$= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \dots + P(A \mid B_n)P(B_n)$$

testing for disease example.



P(both aces | have ace)

P(both aces | have ace of spades)

common errors

$P(A|B) \neq P(B|A)$

$P(A)$ – prior, not $P(A|B)$ posterior

given A occurs implies $P(A|A) = 1$ not $P(A) = 1$

confusing independence + conditional independence.

$P(A \cap B|C) = P(A|C)\,P(B|C)$

opponent of unknown strength

Monty Hall problem.

tree

LOTP

Bayes

Gambler's Ruin

Random walk

$p_i = P(\text{A wins game} \mid \text{A has } \$i)$.

$p_i = p \times p_{i+1} + q \times p_{i-1}$ LOTP

$p_0 = 0$

$p_N = 1$

$$p_i = \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} \qquad q \neq p$$

$$= \frac{i}{N} \qquad q = p$$

Random Variables

function from sample space → real line

RV X

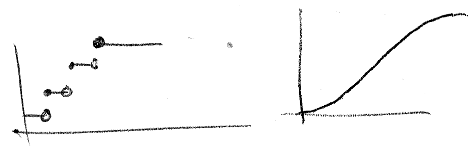
realization x

Bernoulli RV $\quad P(X=1)=p \quad P(X=0)=1-p$

Binomial(n,p) $\quad \underbrace{P(X=k \mid n,p)}_{\text{PMF}} = \binom{n}{k} p^k (1-p)^{n-k}$

– sum of n iid Bernoulli(p).

CDF.



event $X \le x$

$F(x) = P(X \le x)$

increasing
right continuous
$F(x) \to 0 \quad x \to -\infty$
$\to 1 \quad x \to \infty$

$P(a < X \le b) = F(b) - F(a)$.

PMF. – fn that gives prob of RV taking each of allowed values.

$P(X = a_j)$ all j

$p_j \ge 0$

$\sum_j p_j = 1$

Sum of 2 Binomials.

– thinking about # trials.

– thinking about sums of Bernoulli

– manipulating PMFs.

LOTP $\quad \sum_{j=0}^{k} P(X+Y=k \mid X=j)\, P(X=j)$

Hypergeometric – sampling without replacement

$$P(X=k \mid w,b) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$$

Independence of RVs.

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$$ CDF.

discrete $P(X=x, Y=y) = P(X=x)P(Y=y)$ PMF.

Expectations $E(X) = \sum_x x P(X=x)$

linearity

$E(X+Y) = E(X) + E(Y)$

even if not independent

$E(cX) = cE(X)$

indicator RVs. $X = \begin{cases} 1 & A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

$$E(X) = P(A).$$

Binomial $E(X) = np$ via sum of Bernoullis

Hypergeometric. $E(X) = E(X_1 + \dots + X_n)$ n draws

$$= E(X_1) + \dots E(X_n)$$

$$= np$$

Geometric(p). # failures before 1st success.

$$P(X=k) = q^k p$$

$$E(X) = \frac{q}{p}$$

directly: use $\sum_{k=0}^{\infty} q^k = (1-q)^{-1}$

take deriv wrt q.
mult by q.

$$\sum_{k=0}^{\infty} kq^k = \frac{q}{(1-q)^2}$$

$E(X) = p \times 0 + q(1 + E(X))$
(consider 1st flip)
solve for $E(X)$

linearity of Expectation.

$$E[X] = \sum_x x P(X=x)$$

$$= \sum_s X(s) P(\{s\})$$

$T = X + Y$ $E[T] = \sum_s (X+Y)(s) P(\{s\}) = \sum_s X(s) P(\{s\}) + \sum_s Y(s) P(\{s\})$

$$= E[X] + E(Y).$$

Negative Binomial.

\# failures before r^{th} success.

$$P(X=n \mid r,p) = \binom{n+r-1}{r-1} p^r (1-p)^n.$$

$E(X)$ by sum of r Geometric (p)

$$\Rightarrow \frac{rq}{p}$$

1st success distribution.

$X \sim FS(p)$

$Y = X-1 \sim$ Geometric(p)

$E(X) = E(Y)+1 = \frac{q}{p}+1 = \frac{1}{p}.$ — intuitive.

[St Petersburg Paradox]

Poisson.

$$P(X=k \mid \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$E(X) = \lambda$

large # trials, each with small prob of success

Poisson Approx

\# A_js that occur is approx Poisson(λ) with $\lambda = \sum_{j=1}^{n} p_j$

derive as limit of Binomial as $n \to \infty$, $p \to 0$ s.that $np = \lambda$

Continuous Distributions

PDF. $f_X(x)$ if $P(a \le X \le b) = \int_a^b f_X(x)\,dx$. $\forall a,b$

valid: $f_X(x) \geq 0$

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

CDF. $F(x) = P(X \le x) = \cancel{\int_{-\infty}^{x} f_X(x)\,dt} = \int_{-\infty}^{x} f_T(t)\,dt$

PDF. $f_X(x) = F'(x)$.

Variance

$$\text{Var}(X) = E\left((X - E(X))^2 \right)$$

$$= E(X^2) - (E(X))^2$$

Continuous Uniform Dist

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b) \\ 1 & x > b. \end{cases}$$

$$E(X) = \frac{b+a}{2}$$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$ → LOTUS. $E(g(x)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$.

uniform $E(X^2) = \frac{1}{3}$

$\Rightarrow \text{Var}(X) = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{2}$

universality

$U \sim \text{unif}(0,1)$.

$X = F^{-1}(u)$

$X \sim F$

independence.

$P(x_1, x_2, x_3, \dots) = P(x_1)P(x_2)P(x_3)\dots$

$P(X_1 \le x_1, X_2 \le x_2 \dots) = P(X_1 \le x_1)P(X_2 \le x_2)\dots$

Normal Dist.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

$E(z) = 0$

$\text{var}(z) = 1$

$\Phi(z)$ CDF of $N(0,1)$.