

Name: _____ Section: (day/time) _____

AMS131-01 - MIDTERM
Thursday 8th May, 2014.

- You must explain all answers and/or show working for full credit.
- This exam is closed book, but you may use one 8.5 by 11 piece of paper with notes, and a calculator.
- This exam is to be completed individually.

1. [9 points] Prove that

(a)

$$n(n-1) \binom{n-2}{k-2} = k(k-1) \binom{n}{k}$$

(b)

$$\binom{n}{h} \binom{n-h}{k} = \binom{n}{k} \binom{n-k}{h}$$

(c)

$$\sum_{j=1}^n \binom{n}{j}^2 = \binom{2n}{n}$$

2. [4 points] So far we have covered 10 topics in class. I chose 6 of them to be on the exam. You revised 5 topics. You'll get a decent score if 3 of the topics you revised come up on the exam. What's the probability that exactly 3 of the topics you revised will come up on the exam?

3. [13 points] Emails arrive at a rate λ per hour. Let T be the random variable that is the time of arrival of the first email.

(a) By modeling the email arrivals as a Poisson process, and by considering $P(T \geq t)$, derive the CDF of T , and hence show that the pdf of the waiting time is $f_T(t) = \lambda e^{-\lambda t}$, $t > 0$.

(b) Show that the expected waiting time is $1/\lambda$.

(c) Show that the variance of the waiting time is $1/\lambda^2$.

(d) You are waiting for an email, and have already waited time t_0 . What is the pdf of the *additional* waiting time until the next email arrives? [Hint: find $P(T \geq t + h | T \geq t)$]

4. [9 points] To go with the Lego Movie, Lego sell minifigures of the characters from the movie. They are sold in packets, where each packet contains one minifigure, and from the outside of the packet it is impossible to tell which minifigure is inside. There are n minifigures to collect.

(a) Assuming that each packet that my son buys is equally likely to contain any one of the minifigures, show that the expected number of packets that he needs to buy to collect the whole set is approximately $n \log n$.

[Hint: express the random variable that is the total number of packets as a sum of simpler terms, and use linearity of expectation.]

- (b) If, in addition to the 16 characters from the movie, there is also a special, gold, character, which is present in one in five hundred of the packets, give an approximate value for the number of packets required to collect the full set together with the gold character. Explain your answer clearly.