33 points total.

Name:____________________________ Section: (day/time) ________________________

AMS131-01 - MIDTERM
Thursday 3rd May, 2018.

- You must explain all answers and/or show working for full credit.
- This exam is closed book, but you may use one 8.5 by 11 piece of paper with notes, and a calculator.
- You may leave answers in terms of Binomial Coefficients, suitably simplified.
- This exam is to be completed individually.

1. [3 points] If k people are seated in a random manner in a circle containing n chairs ($n > k$), what is the probability that the people will occupy k adjacent chairs in the circle?

There are $\binom{n}{k}$ ways of sitting k people in n chairs.

Because the chairs are in a circle, n of these have the k occupied chairs being adjacent.

$\Rightarrow$ prob. is $\frac{n}{\binom{n}{k}}$ $\left(= \frac{(n-k)!\,k!}{(n-1)!} \right)$

2. [2 points] If n letters are placed at random in n envelopes, what is the probability that exactly $n-1$ letters will be placed in the correct envelopes?

if $n-1$ letters are in the correct envelope, then all n are in the correct envelope.

$\Rightarrow$ prob. that <u>exactly</u> $n-1$ are in the correct envelope is <u>zero</u>.

3. [5 points] Three players, A, B, C take turns tossing a fair coin. Suppose that A tosses first, B tosses second, and C tosses third; and suppose that this cycle is repeated indefinitely until someone wins by being the first player to obtain a head. Determine the probability that each player will win.

For A to win, either they toss a H, or ~~B and C~~ they toss a T, and B + C both toss T, then the situation is as at the start. i.e

$$P(A) = \frac{1}{2} + \left(\frac{1}{2}\right)^3 \times P(A).$$

$$P(A) = \frac{1}{2} + \frac{1}{8} P(A) \quad \Rightarrow \quad P(A) = \frac{4}{7}.$$

For B to win, (A throws T and B throws H) or (all three throw T and we restart).

$$P(B) = \frac{1}{2} \times \frac{1}{2} + \left(\frac{1}{2}\right)^3 P(B) \qquad P(B) = 2/7$$

$$P(C) = 1 - \frac{4}{7} - \frac{2}{7} = \frac{1}{7}$$

4. [3 points]

(a) A fair die is rolled. Find the expected value of the roll.

$$E\{X\} = \sum x P(X=x) = (1+2+3+4+5+6) \times \frac{1}{6} = 3.5$$

(b) Four fair dice are rolled. Find the expected total of the rolls.

$Y = X_1 + X_2 + X_3 + X_4$ where each X is as in part (a).

$$E(Y) = E(X_1 + X_2 + X_3 + X_4) = \cancel{4}.$$

$$= 4E(X_1)$$

$$= 4 \times 3.5$$

$$= 14$$

5. [5 points] In one of the homework problems, we considered a mixture distribution, where $F1$ and $F2$ are CDFs, $0 < p < 1$, and $F(x) = pF1(x) + (1-p)F2(x)$.

   (a) what is the expected value of a random variable that follows $F(x)$?

$$f(x) = \frac{d}{dx} F(x) = p f_1(x) + (1-p) f_2(x)$$

$$E(x) = \int x f(x)\, dx = p \int x f_1(x)\, dx + (1-p) \int x f_2(x)\, dx$$

$$= p\, E_1(x) + (1-p)\, E_2(x).$$

There are times when using the Poisson distribution to model the probabilities of the number of events underestimates the probability of zero events. In these cases a *zero inflated* Poisson distribution is used, which has PMF

$$P(Y=0) = p + (1-p)e^{-\lambda} \quad (1)$$

$$P(Y=k) = (1-p)\frac{\lambda^k e^{-\lambda}}{k!}, k > 0 \quad (2)$$

where $0 < p < 1$.

   (b) What is the expected value of Y

Comparing to part (a), we have that distribution 1 is a discrete PMF with $P(X=0) = 1$, and distribution 2 is a Poisson.

$$\Rightarrow E(x) = p \times 0 + (1-p)\lambda$$

$$= (1-p)\lambda.$$

6. [2 points] Let X be a discrete random variable with support $-n, -n+1, \ldots, 0, \ldots, n-1, n$ for some positive integer n. Suppose that the PMF of X satisfies the symmetry property $P(X=-k) = P(X=k)$ for all integers k. Find $E(X)$.

$$E[x] = \sum x\, P(x=x)$$

Because the support is symmetric, and $P(x=-k) = P(x=k)$, the expected value is zero.

7. [6 points] Suppose that the random varibles $X_1, \ldots, X_n$ form n Bernoulli(p) trials. Determine the conditional probability that $X_1 = 1$ given that

$$\sum_{i=1}^{n} X_i = k \quad (k = 1, \ldots, n).$$

$$P(X_1 = 1 \mid \sum x_i = k) = \frac{P(X_1 = 1 \text{ and } \sum_{i=1}^{n} x_i = k)}{P(\sum_{i=1}^{n} x_i = k)}$$

$$= \frac{P(X_1 = 1) \times P(\sum_{i=2}^{n} x_i = k-1)}{P(\sum_{i=1}^{n} x_i = k)} = \frac{P(X_1 = 1)\, P(k-1 \text{ successes in } n-1 \text{ Bernoulli}(p) \text{ trials})}{P(k \text{ successes in } n \text{ Bernoulli}(p) \text{ trials})}$$

$$= \frac{p \times \binom{n-1}{k-1} p^{k-1} (1-p)^{n-1-(k-1)}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

8. [4 points]

(a) Independent Bernoulli trials are performed, with probability 1/2 of success, until there has been at least one success. Find the PMF of the number of trials performed.

X - number of trials

for $X = n$, $(n-1)$ failures followed by 1 success

$$P(X = n) = \left(\tfrac{1}{2}\right)^{n-1} \times \left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^{n}$$

(b) Independent Bernoulli trials are performed, with probability 1/2 of success, until there has been at least one success and at least one failure. Find the PMF of the number of trials performed.

Y - number of trials, for $Y = n$

either: $(n-1)$ failures followed by 1 success

or $(n-1)$ successes followed by 1 failure

$$P(Y = n) = \left(\tfrac{1}{2}\right)^{n-1} \times \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^{n-1} \times \tfrac{1}{2} = \left(\tfrac{1}{2}\right)^{n-1} \qquad n \geq 2.$$

$$\left[P(Y=n) = P(Y=n \mid A) P(A) + P(Y=n \mid A^c) P(A^c) \right.$$

$$\left. \text{where } A \equiv 1^{st} \text{ trial is } \cancel{\text{succ}} \text{ failure.} \right]$$

9. [3 points] A continuous random variable is said to have the *Exponential Distribution* with parameter λ, where $\lambda > 0$, if its PDF is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

and CDF is

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0.$$

The Exponential Distribution is often used to model *waiting times*, i.e. $f(x)$ is the PDF for the waiting time x, until something happens.

Using the definition of conditional probability, show that the Exponential Distribution has the *Memoryless Property*, i.e. that

$$P(X \geq s + t | X \geq s) = P(X \geq t).$$

(i.e., if you have already waited time s, the probability that you will have to wait another t minutes is exactly the same as the probability of having to wait t minutes if you haven't waited at all.)

$$P(x \geq s+t | x \geq s) = \frac{P(x \geq s+t \text{ and } x \geq s)}{P(x \geq s)}$$

$$= \frac{P(x \geq s+t)}{P(x \geq s)}$$

$$= \frac{1 - F(s+t)}{1 - F(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda(s)}} = e^{-\lambda t}. = P(x \geq t).$$

2