AMS131-01 - MIDTERM
Thursday 3rd May, 2018.

- You must explain all answers and/or show working for full credit.
- This exam is closed book, but you may use one 8.5 by 11 piece of paper with notes, and a calculator.
- You may leave answers in terms of Binomial Coefficients, suitably simplified.
- This exam is to be completed individually.

1. [3 points] If \( k \) people are seated in a random manner in a circle containing \( n \) chairs \((n > k)\), what is the probability that the people will occupy \( k \) adjacent chairs in the circle?

There are \( \binom{n}{k} \) ways of sitting \( k \) people in \( n \) chairs.

Because the chairs are in a circle, \( n \) of them have the \( k \) occupied chairs being adjacent.

\[
\Rightarrow \text{Prob.} = \frac{n}{\binom{n}{k}} = \frac{(n-k)!k!}{(n-1)!}
\]

2. [2 points] If \( n \) letters are placed at random in \( n \) envelopes, what is the probability that exactly \( n-1 \) letters will be placed in the correct envelopes?

If \( n-1 \) letters are in the correct envelopes, then all \( n \) are in the correct envelopes.

\[
\Rightarrow \text{Prob. that exactly } n-1 \text{ are in the correct envelope is zero.}
\]
3. [5 points] Three players, A, B, C take turns tossing a fair coin. Suppose that A tosses first, B tosses second, and C tosses third; and suppose that this cycle is repeated indefinitely until someone wins by being the first player to obtain a head. Determine the probability that each player will win.

For A to win, either they lose a H, or B and C both lose T, with the structure:

\[ P(A) = \frac{1}{2} + \left(\frac{1}{2}\right)^3 \times P(A). \]

\[ P(A) = \frac{1}{2} + \frac{1}{8} P(A) \quad \Rightarrow \quad P(A) = \frac{4}{7}. \]

For B to win, (A throws T and B throws H) or (all three throw T and we restart).

\[ P(B) = \frac{1}{2} \times \frac{1}{2} + \left(\frac{1}{2}\right)^3 \times P(B) \quad \Rightarrow \quad P(B) = \frac{2}{7}. \]

\[ P(C) = 1 - \frac{4}{7} - \frac{2}{7} = \frac{1}{7}. \]

4. [3 points]

(a) A fair die is rolled. Find the expected value of the roll.

\[ E[X] = \sum x \times P(X=x) = (1 + 2 + 3 + 4 + 5 + 6) \times \frac{1}{6} = 3.5 \]

(b) Four fair dice are rolled. Find the expected total of the rolls.

\[ Y = x_1 + x_2 + x_3 + x_4 \quad \text{where each } x_i \text{ is an outcome of roll } i. \]

\[ E(Y) = E(x_1 + x_2 + x_3 + x_4) = 4 \times E(x_1) \]

\[ = 4 \times 3.5 \]

\[ = 14 \]
5. [5 points] In one of the homework problems, we considered a mixture distribution, where \( F_1 \) and \( F_2 \) are CDFs, \( 0 < p < 1 \), and \( F(x) = p F_1(x) + (1 - p) F_2(x) \).

(a) What is the expected value of a random variable that follows \( F(x) \)?

\[
\mathbb{E}(x) = \int x \, f(x) \, dx = p \int x f_1(x) \, dx + (1 - p) \int x f_2(x) \, dx
= p \, \mathbb{E}_1(x) + (1 - p) \, \mathbb{E}_2(x).
\]

There are times when using the Poisson distribution to model the probabilities of the number of events underestimates the probability of zero events. In these cases a zero inflated Poisson distribution is used, which has PMF

\[
P(Y = 0) = p + (1 - p)e^{-\lambda} \quad (1)
\]

\[
P(Y = k) = (1 - p) \frac{\lambda^k e^{-\lambda}}{k!}, \quad k > 0 \quad (2)
\]

where \( 0 < p < 1 \).

(b) What is the expected value of \( Y \)?

Comparing to part (a), we have that distribution 1 is a discrete PMF with \( P(X = 0) = 1 \), and distribution 2 is a Poisson.

\[
\Rightarrow \mathbb{E}(X) = p \times 0 + (1 - p) \lambda
= (1 - p) \lambda.
\]

6. [2 points] Let \( X \) be a discrete random variable with support \(-n, -n + 1, \ldots, 0, \ldots, n - 1, n\) for some positive integer \( n \). Suppose that the PMF of \( X \) satisfies the symmetry property \( P(X = -k) = P(X = k) \) for all integers \( k \). Find \( E(X) \).

\[
E[X] = \sum x \, p(x = x)
\]

Because the support is symmetric, and \( p(x = -k) = p(x = k) \),

the expected value is \( \boxed{0} \).
7. [6 points] Suppose that the random variables $X_1, \ldots, X_n$ form $n$ Bernoulli($p$) trials. Determine the conditional probability that $X_1 = 1$ given that

$$\sum_{i=1}^{n} X_i = k \quad (k = 1, \ldots, n).$$

$$P(X_1 = 1 | \sum_{i=1}^{n} X_i = k) = \frac{P(X_1 = 1 \text{ and } \sum_{i=2}^{n} X_i = k-1)}{P(\sum_{i=1}^{n} X_i = k)}.

= \frac{P(X_1 = 1) \cdot P\left(\sum_{i=2}^{n} X_i = k-1\right)}{P\left(\sum_{i=1}^{n} X_i = k\right)} = \frac{p \cdot (n-1) \cdot p^{k-1} \cdot (1-p)^{n-1-(k-1)}}{\binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}} = \frac{n-1 \cdot p^{k-1} \cdot (1-p)^{n-1-k}}{n \cdot \binom{n}{k}} = \frac{k}{n}.

8. [4 points]

(a) Independent Bernoulli trials are performed, with probability $1/2$ of success, until there has been at least one success. Find the PMF of the number of trials performed.

$$X - \text{number of trials}$$

$$\mathbb{P}(X = n) = (\frac{1}{2})^{n-1} \cdot \frac{1}{2} = (\frac{1}{2})^n.$$

(b) Independent Bernoulli trials are performed, with probability $1/2$ of success, until there has been at least one success and at least one failure. Find the PMF of the number of trials performed.

$$Y - \text{number of trials, for } Y = n$$

$$\mathbb{P}(Y = n) = (\frac{1}{2})^{n-1} \cdot \frac{1}{2} + (\frac{1}{3})^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1} \quad n \geq 2.$$
9. [3 points] A continuous random variable is said to have the *Exponential Distribution* with parameter \( \lambda \), where \( \lambda > 0 \), if its PDF is

\[
f(x) = \lambda e^{-\lambda x}, \quad x > 0
\]

and CDF is

\[
F(x) = 1 - e^{-\lambda x}, \quad x > 0.
\]

The Exponential Distribution is often used to model *waiting times*, i.e. \( f(x) \) is the PDF for the waiting time \( x \), until something happens.

Using the definition of conditional probability, show that the Exponential Distribution has the *Memoryless Property*, i.e. that

\[
P(X \geq s + t | X \geq s) = P(X \geq t).
\]

(i.e., if you have already waited time \( s \), the probability that you will have to wait another \( t \) minutes is exactly the same as the probability of having to wait \( t \) minutes if you haven’t waited at all.)

\[
P(X \geq s + t | X \geq s) = \frac{P(X \geq s + t \text{ and } X \geq s)}{P(X \geq s)}
\]

\[
= \frac{P(X \geq s + t)}{P(X \geq s)}
\]

\[
= \frac{1 - F(s + t)}{1 - F(s)} = \frac{e^{-\lambda(s + t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \geq t).
\]