robability.

3 ~~ke~~ views of probability

frequency
classical
Bayesian / subjective

Experiments
events.

$$P(A) = \frac{\#\ \text{favourable outcomes}}{\#\ \text{possible outcomes}}$$

- counting
- Binomial coefficient $\binom{n}{k}$

ways of choosin
k items out of n
whose order does
not matter.

chose k out of n.

	order matters	order does not matter
with replacement	n^k	?
without replacement	$n \times (n-1) \times (n-2) \ldots \times (n-k+1)$	$\binom{n}{k}$

How many ways are there to pick k times from a set of n objects, where order does not matter, with replacement?

$k = 0$ # ways $= 1$

$k = 1$ n

~~$k=2$~~

$= 2.$

$k = 7$

How many ways can I split k indistinguishable objects between 2 boxes.

ways $= k+1$

eneralize.

$n = 5$

$k = 8$

Distribute k items into n boxes.

So, how many ways are there to put k items into n boxes?

$\circ \; \circ \;|\; \circ \; \circ \; \circ \;|\;|\; \circ \;|\; \circ \; \circ$

k objects

$n-1$ separators

ways to arrange these $(k+n-1)$ "objects"

is the # of ways to put k items into n boxes.

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

$\binom{n}{k} = \binom{n}{n-k}$

$\binom{k+n-1}{k}$: pick where to put the k objects

$\binom{k+n-1}{n-1}$: pick where to put the $(n-1)$ separators.

$k=0 \qquad \binom{n-1}{0} = 1$

$k=1 \qquad \binom{n}{k} = n$

$n=2 \qquad \binom{k+1}{k} = k+1$

Birthday Problem.

For a group of n people, what's the probability that no two of them share a birthday?

$P(\text{no two share a birthday}) = 0 \qquad n \geq 366$

$n \leq 365 \qquad \dfrac{365 \times 364 \times 363 \times \dots \times (365 - n + 1)}{365^n}$

$P(\text{match}) = 1 - P(\text{no two share a birthday})$

How large must n be so that $P(\text{match}) \simeq 0.5$?

$n = 23 \qquad P(\text{match}) \simeq 50.7\%$

$n = 100 \qquad P(\text{match}) > 99.99\%$

The important quantity is the number of ~~pairs~~ pairs of people.

$\binom{n}{2}$ pairs

$\binom{23}{2} = \dfrac{23 \times 22}{2} = 253$

Non Naive Definition of Probability

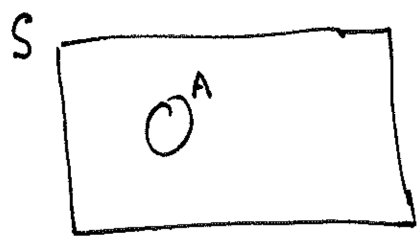
- not all outcomes are equally likely
- not be finitely many outcomes.

Probability space.

S - sample space

P - function which takes an event

$A \subseteq S$ and returns $P(A) \in [0,1]$

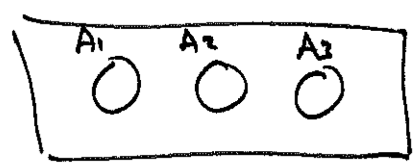


$P(\phi) = 0$

$P(S) = 1$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

if A_n are disjoint.



Properties

1/ $P(A^c) = 1 - P(A)$

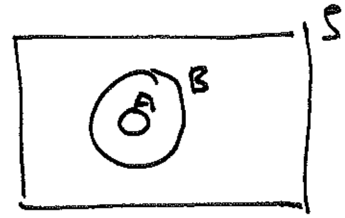
$$1 = P(S) = P(A \cup A^c)$$
$$= P(A) + P(A^c)$$

as A, A^c are disjoint

$$P(A^c) = 1 - P(A)$$

if $A \subseteq B$ $\quad P(A) \le P(B)$

subset



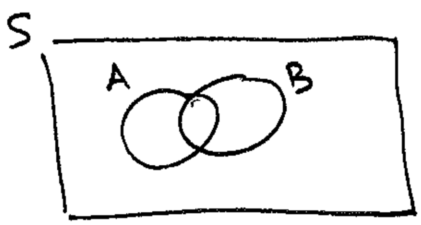
$$B = A \cup \underbrace{(B \cap A^c)}_{\text{the part of } B \text{ that is not in } A}$$

disjoint events

$$P(B) = P(A) + P(B \cap A^c).$$

$$P(B) \geq P(A)$$

$P(A \cup B)$ when A, B are not disjoint?



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

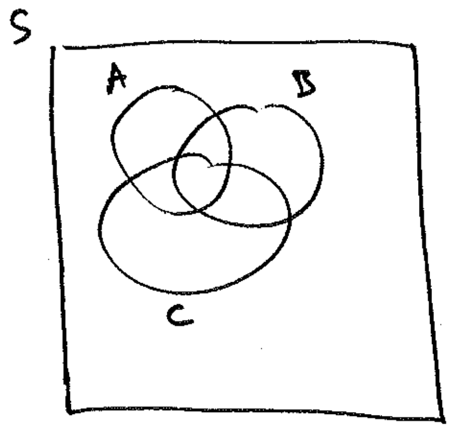
$$P(A \cup B) = \cancel{P(A^c)}\ P(A \cup (B \cap A^c))$$

$$= P(A) + P(B \cap A^c)$$

Require that $\quad P(B) - P(A \cap B) = P(B \cap A^c)$

$$P(B) = P(B \cap A) + P(B \cap A^c).$$

several $P(A \cup B \cup C)$



$$P(A) + P(B) + P(C)$$
$$- P(A \cap B)$$
$$- P(A \cap C)$$
$$- P(B \cap C)$$
$$+ P(A \cap B \cap C).$$

Generalize to n events.

$$P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n)$$

$$= \sum_{j=1}^{n} P(A_j)$$

$$- \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$-$$
$$\vdots$$

$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap A_3 \cdots \cap A_n)$$

Example.

n letters, n envelopes

put each letter in a randomly chosen envelope

what's the chance that at least one letter ends up in the correct envelope?

A_j - the letter for person j ends up in the j^{th} envelope.

$$P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n)$$

$$= \sum_{j=1}^{n} P(A_j) \qquad n \times \frac{1}{n}$$

$$- \sum_{i<j} P(A_i \cap A_j) \qquad \binom{n}{2} \frac{1}{n} \frac{1}{n-1}$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \qquad \binom{n}{3} \times \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$$

$-$

$+$

$-$

$\vdots$

$$n \times \frac{1}{n} = 1$$

$$\binom{n}{2} \frac{1}{n} \times \frac{1}{n-1} = \frac{n!}{(n-2)!\,2!} \times \frac{1}{n} \times \frac{1}{n-1}$$

$$= \frac{n \times n-1}{2!} \times \frac{1}{n} \times \frac{1}{n-1}$$

$$= \frac{1}{2!} = \frac{1}{2}$$

$$\binom{n}{3} \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} = \frac{n!}{(n-3)!\,3!} \times \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$$

$$= \frac{1}{3!}$$

$$P(A_1 \cup A_2 \cup \ldots A_n) = 1 \underbrace{- \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \cdots}_{}$$

Taylor series for e^{x}
evaluated at $x = -1$

$$\simeq 1 - \frac{1}{e} \qquad \text{(as } n \text{ gets large).}$$

$$P(\text{no one gets the right letter}) \approx \frac{1}{e} \simeq 0.37$$