Review.

Probability

① - frequency
② - classical - equally likely outcomes.
③ - subjective

State space S

for ② every element of S is equally likely.

Probability is defined on <u>events</u> $A \subseteq S$

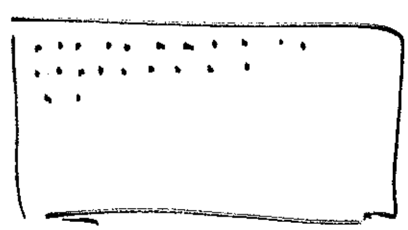
$P(A)$.

$$P(A) = \frac{\#\text{ elements of } S \text{ that are in } A.}{\#\text{ elements of } S}$$

A = Poker hand is a full house.

3 cards of one type (A, K, Q, J ... 3, 2).
2 cards of another type.

$$P(A) = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}}$$



20 boxes.

12 balls.

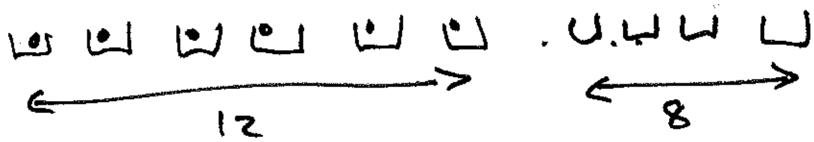
Throw balls randomly into boxes.

Prob. that no box will receive more than one ball.

$$\frac{20}{20} \times \frac{19}{20} \times \frac{18}{20} \times \frac{17}{20} \times \cdots \times \frac{9}{20}$$

$$\frac{20!}{20^{12}.\ 8!} = \frac{20!}{8!\ 20^{12}}$$

# of ways of putting 12 balls into 20 boxes such that no box has more than one ball.



$\binom{20}{12}$.

$$\frac{\binom{20}{12}}{20^{12}}$$

Count the number of ways of getting 12 boxes with balls & 8 boxes without.

$$20 \times 19 \times \cdots \times 9.$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Non nieve definition.

S - sample space.

P - function that takes an event A and returns $P(A) \in [0,1]$

$$P(\emptyset) = 0$$

$$P(S) = 1.$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \qquad A_i \text{ disjoint}$$

$P(A^c) = 1 - P(A)$

often, if asked "find the probability that at least ....."

eg find prob of at least one six on 5 rolls of a die.

A^c = no sixes on 5 rolls

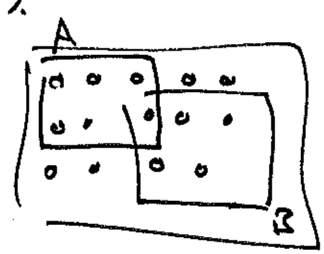
$$P(A^c) = \left(\frac{5}{6}\right)^5$$

$$\Rightarrow P(A) = 1 - \left(\frac{5}{6}\right)^5$$

$A \subseteq B \quad P(A) \leq P(B)$

union of events that may not be disjoint

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



inclusion-exclusion to generalize to more than 2 events.

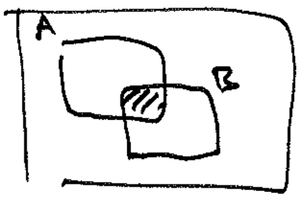
Independence.

$$P(A \cap B) = \cancel{P(B} \; P(A)\,P(B)$$

conditional probability.

Two events A, B.

$P(A|B)$ prob. that event A happens, knowing that B has happened



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

what to condition on?

$X_1 \sim \text{Bin}(n, p)$.

$X_2 \sim \text{Bin}(m, p)$ independent.

$$P(X_1 \mid \underbrace{X_1 + X_2 = k}_{})$$

extra information.

changes the distribution for X_1 - it is no longer Binomial.

$$P(x_1 \mid x_1 + X_2 = k) = \cancel{\phi}$$

$$P(\underbrace{X_1 = x_1}_{A} \mid \underbrace{X_1 + X_2 = k}_{B})$$

$$\frac{P(X_1 = x_1 \text{ and } X_1 + \cancel{X_2} = k)}{P(X_1 + X_2 = k)}$$

$$\frac{P(X_1 = x_1 \text{ and } X_2 = k - \overset{x_1}{\cancel{X_1}})}{P(X_1 + X_2 = k)}$$

$$\frac{P(X_1 = x_1) \times P(X_2 = k - \overset{x_1}{\cancel{X_1}})}{P(X_1 + X_2 = k)}$$

$$\frac{\binom{n}{x_1} p^{x_1} (1-p)^{n-x_1} \binom{M}{k-x_1} p^{k-x_1} (1-p)^{M-k+x_1}}{\binom{M+n}{k} p^{k} (1-p)^{M+n-k}}$$

$$\boxed{\frac{\binom{n}{x_1}\binom{M}{k-x_1}}{\binom{M+n}{k}}}$$

Hypergeometric.

careful choice of the event that you condition on can be the key to solving a problem.

$F1(x)$

$F2(x)$

generate a Bernoulli (p) RV.

if its a success, draw X from F1

failure F2.

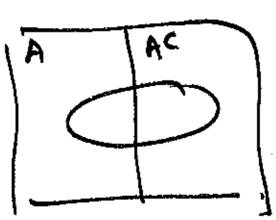
what's the CDF of X ?

let A be the event that the Bernoulli RV is a success.

$$F(x) = P(\underbrace{X \leq x}_{\text{event}})$$

LOTP

$$P(B) = P(B \mid A) P(A) + P(B \mid A^c) P(A^c)$$



B - $X \leq x$

A - success.

$$P(X \leq x) = P(X \leq x \mid \text{success}) . P(\text{success}) + P(X \leq x \mid \text{failure}) . P(\text{failure}).$$

$$= F_1(x) \times p + F_2(x) \times (1-p)$$

$$\cancel{\not} \quad F(x) = p F_1(x) + (1-p) F_2(x).$$

- think about the <u>events</u>.
- think about conditioning $\longleftrightarrow$ LOTP.
-

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c). \Longleftarrow$$

$$= P(B \cap A) + P(B \cap A^c)$$

$$= P(B).$$

A, B, D are 3 events.

$$P(A|D) \geq P(B|D).$$

$$P(A|D^c) \geq P(B|D^c).$$

what can we say about ~~P(A) and~~ the relationship between P(A) and P(B) ?

$$P(A) = P(A|D)P(D) + P(A|D^c)P(D^c)$$

$$\geq P(B|D)P(D) + P(B|D^c)P(D^c).$$

$$\geq P(B).$$

Prove that

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

algebra

or explain how they're both counting the same thing.

Bayes theorem.

$$P(B \mid A) = \frac{P(A \mid B)\, P(B)}{P(A)}.$$

Reversing the order of the conditioning

3 factories manufacture widgets.

A	makes 20% of all widgets,	5% of which are faulty
B	50%	10%
C	30%	20%

You have a faulty widget, what's the chance that it came from factory C?

$P(\text{came from } C \mid \text{faulty})$

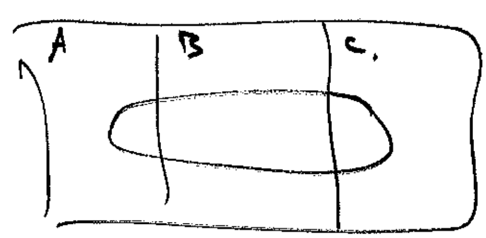
$$= \frac{P(\text{faulty} \mid C)\, P(C)}{\cancel{P(C)}\; P(\text{faulty})}$$

$$= \frac{P(\text{faulty} \mid C)\, P(C)}{P(\text{faulty} \mid A) P(A) + P(\text{faulty} \mid B) P(B) + P(\text{faulty} \mid C) P(C)}$$

$$= \frac{0.2 \times 0.3}{0.05 \times 0.2 + 0.1 \times 0.5 + 0.2 \times 0.3}$$

$$= \underline{\underline{0.5}}$$

If we have a faulty widget, 50% of the time it will have come from factory C.



Random Variables.

Function from state space $\longrightarrow$ real line.

RV X

realization x

Bernoulli (p). $\quad P(X=1) = p \qquad P(X=0) = (1-p)$

Binomial (n, p). $\quad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

Hypergeometric (w, b, n). $\quad = \dfrac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$

Expected value.

$$E[x] = \sum x_i P(x_i).$$

$$\sum x P(X=x)$$

Bernoulli $\quad - \; E(X) = p \qquad \leftarrow$ Indicator Variable.

Binomial $\quad np$

Hypergeometric $\quad np$

Indicator variable for an event A.

Expected value $\equiv$ prob. of event A.

$\text{Bin}(n, p) \equiv$ sum of n Bernoulli (p) RVs.

$$E(\text{Bin}) = E(\text{sum of } n \text{ Bernoulli } (p) \text{ RV}).$$

$$E(X) = E(X_1 + X_2 + X_3 \ldots X_n)$$

Linearity of Expectation

$$= E(X_1) + E(X_2) + \ldots E(X_n)$$

$$= p + p + \cdots + p$$

$$= np$$

Each Happy meal comes with a toy.

There are n different toys.

How many Happy Meals do you expect to have to eat to collect the whole set?

(each toy has prob. $\frac{1}{n}$ to be in a given meal)

Y - total number of happy meals eaten.

$$Y = X_1 + X_2 + X_3 + \dots X_n$$

additional number of meals to get the i^{th} toy.

$P(X_i = x_i)$ = Prob of getting $(x_i - 1)$ "failures" followed by a success.

$$P(X_2 = x) = \left(\frac{1}{n}\right)^{x-1} \frac{n-1}{n}$$

$$P(X_3 = x) = \left(\frac{2}{n}\right)^{x-1} \frac{n-2}{n}$$

$$P(X_i = x) = \left(\frac{i-1}{n}\right)^{x-1} \frac{n-i}{n}$$

First success distribution.

$$E[Y] = E[X_1] + E[X_2] + \dots E[X_n]$$

$$= 1$$

expected value of the different FS distributions

$$= 1 + \frac{n}{n-1} + \frac{n}{n-2} + \frac{n}{n-3} + \dots n$$

$$E[Y] \simeq n \log n.$$

Poisson Distribution.

$$P(X = k \mid \lambda) = \frac{e^{-\lambda} \lambda^{k}}{k!}$$

$$E(X) = \lambda.$$

Continuous Distributions.

PDF $\quad f(x) \quad$ Prob. $\displaystyle\int_A f(x)\, dx.$

$$E(x) = \int x f(x)\, dx$$

$$E[g(x)] = \int g(x) f(x)\, dx$$