uniform distribution

$U \sim \text{Unif}(0, 1)$

$X \sim F(x)$

let $x = F^{-1}(U)$.

then $X \sim F(x)$

useful when we want to generate random numbers with a particular distribution.

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if we have a complicated function $g(x)$ we may not be able to compute

$$E[g(x)] = \int g(x) f(x)\, dx$$ analytically.

can compute the expectation by

generating ~~and~~ realizations x_i of $f(x)$

computing $g(x_i)$

averaging the $g(x_i)$ values.

Monte Carlo Method.

$$F(x) = 1 - e^{-x} \quad x > 0$$

$$F^{-1}(u) = -\log(1-u)$$

Simulate $u \sim \text{Unif}(0,1)$

$$x = -\log(1-u)$$ ← these values are distributed as $\text{Exp}(1)$

exponential distribution with $\lambda = 1$

<u>Independence.</u>

$X_1, \ldots X_n$ are independent if.

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots X_n \leq x_n)$$ joint CDF.

$$= P(X_1 \leq x_1) P(X_2 \leq x_2) \ldots P(X_n \leq x_n).$$ product of marginal CDFs

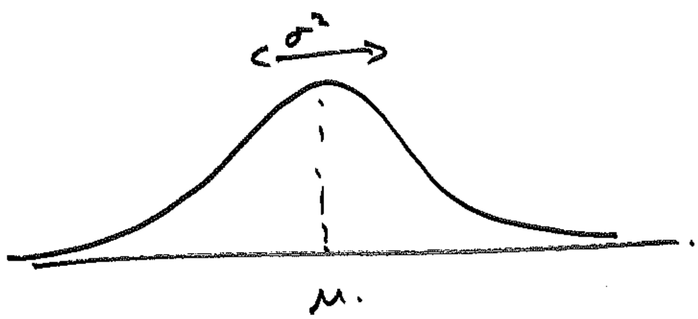
for all $x_1, x_2, \ldots x_n$

The pdf will factorize in a similar manner.

Normal Distribution

- Important due to the Central Limit Theorem

CLT - the sum of a large number of iid random variables with any (finite variance) distribution will be approximately Normal.



$N(\mu, \sigma^2)$.

mean, variance

Standard Normal $N(0,1)$

pdf $f(z) = c\, e^{-z^2/2}$

$= c \exp(-z^2/2)$.

— symmetric about 0

tends to 0 as $z \to \pm\infty$

what is c?

Recall - to be a valid pdf

$$\int_{-\infty}^{\infty} f(z)\, dz = 1$$

$$\Rightarrow \quad c = \frac{1}{\int_{-\infty}^{\infty} \exp(-z^2/2)\, dz}$$

$$\frac{1}{c} = \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

substitution ✗

by parts ✗

$$I = \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

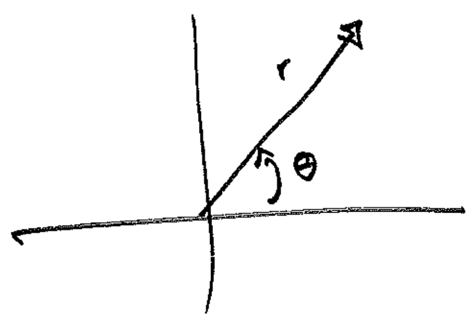
$$I^2 = \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \times \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy$$

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy$$

Transform to polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$



The Jacobian of the transformation relates $\underline{dx\,dy}$ to $dr\,d\theta$

small area in x-y coordinates

Jacobian $\left|\dfrac{d(x,y)}{d(r,\theta)}\right| = \begin{vmatrix} \dfrac{dx}{dr} & \dfrac{dx}{d\theta} \\ \dfrac{dy}{dr} & \dfrac{dy}{d\theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r)\sin^2\theta.$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

hence $dx\,dy \longrightarrow r\,dr\,d\theta$

$$I^2 = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2}\, r\, dr\, d\theta.$$

let $u = \frac{r^2}{2}$

$du = r\,dr$

$$I^2 = \int_{\theta=0}^{2\pi} \int_{u=0}^{\infty} e^{-u} \, du \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(-e^{-u} \Big|_0^{\infty} \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} 1 \, d\theta = \underline{\underline{2\pi}}$$

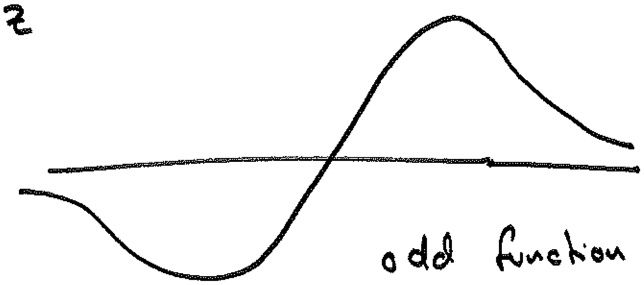
$I^2 = 2\pi$ $\qquad$ $C = \frac{1}{\sqrt{2\pi}}$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \underline{\underline{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}}$$

Mean + Variance of Standard Normal.

$$E[z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \, e^{-z^2/2} \, dz$$

symmetric



odd function

$$= 0$$

$$\text{Var}[z] = E[z^2] - (E[z])^2$$

$$E[z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \, e^{-z^2/2} \, dz.$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 \, e^{-z^2/2} \, dz$$

integrate by parts

$$\int u \, dv = uv - \int v \, du$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} z \times z \, e^{-z^2/2} \, dz$$

$$u = z \qquad du = dz$$

$$dv = z \, e^{-z^2/2} \qquad v = -e^{-z^2/2}.$$

$$E\left[z^2\right] = \frac{2}{\sqrt{2\pi}}\left[uv\Big]_0^\infty - \int_0^\infty v\, du \right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[-z e^{-z^2/2}\Big|_0^\infty - - \int_0^\infty e^{-z^2/2}\, dz \right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[0 + \frac{1}{2}\sqrt{2\pi} \right]$$

$$\Rightarrow \quad E\left[z^2\right] = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2} = 1$$

Hence $\text{Var}[z] = 1$.

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can't do the definite integrals $\left(\text{eg} \int_{-\infty}^{x} \exp(-x^2/2)\, dx \right)$ analytically.

they can be computed accurately.

$\Phi(z)$ = CDF of ~~standard~~ Normal.

$$\Phi(-z) = 1 - \Phi(z)$$