webcast.

username : ams-131-1

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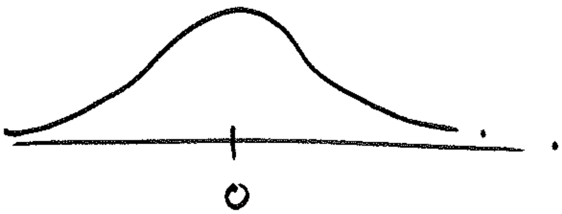
Normal Distribution.



Standard Normal

mean $\mu = 0$

variance $\sigma^2 = 1$

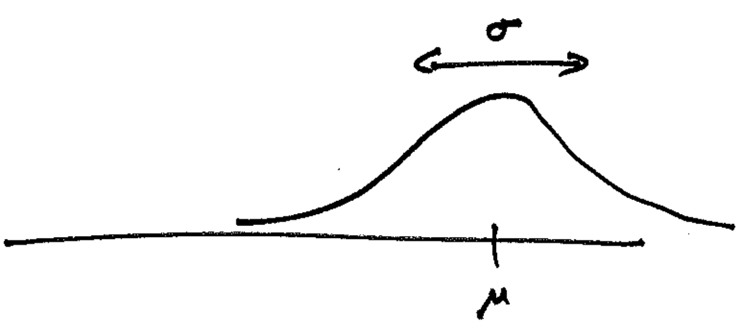
$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

CDF $\Phi(x)$

$$E[X^2] = 1$$

General Normal.

$$X = \mu + \sigma Z \qquad Z \sim N(0,1)$$



μ - mean of X

General.

$Z \sim f_Z(z)$

$X = g(z)$

what is $f_X(x)$.

$$\begin{aligned} E[X] &= E[\mu + \sigma Z] \\ &= E[\mu] + E[\sigma Z] \\ &= \mu + \sigma E[Z] \\ &= \mu \end{aligned}$$

$$\mathrm{Var}[X] = \mathrm{Var}[\mu + \sigma Z]$$

$$\mathrm{Var}[X + c] = E\left[\left(x + c - E(x + c) \right)^2 \right]$$

$$\begin{aligned} \mathrm{Var}[X] &= E\left[(X - E(X))^2 \right] \\ &= E(X^2) - (E[X])^2 \end{aligned}$$

$$E\left[\left(x + c - E(x) - c \right)^2 \right]$$

$$= E\left[(X - E(x))^2 \right]$$

$$= \mathrm{Var}[X]$$

$$\mathrm{Var}[cX] = c^2\, \mathrm{Var}[X]$$

$$\mathrm{Var}[X + Y] \neq \mathrm{Var}[X] + \mathrm{Var}[Y] \quad \text{in general.}$$

$$= \mathrm{Var}[X] + \mathrm{Var}[Y] \quad \text{if } X, Y \text{ are independent.}$$

$$\text{Var}[X + X] = \text{Var}[2X] = 4\,\text{Var}[X].$$

$\text{Var}[\mu + \sigma Z]$ — Variance of General Normal

$$= \text{Var}[\mu] + \text{Var}[\sigma Z]$$

$$= 0 + \sigma^2\,\text{Var}[Z]$$

$$= \sigma^2$$

Hence $X = \mu + \sigma Z$ where $Z \sim N(0,1)$

has mean μ

variance σ^2

$$Z = \cancel{\ } \frac{X - \mu}{\sigma} \quad \leftarrow \text{Standardization.}$$

$\cancel{*}$ $\frac{X-\mu}{\sigma}$ is dimensionless.

PDF of X

$$P(X \le x) = \cancel{*}\ P\Big(\underbrace{\frac{X - \mu}{\sigma}}_{\substack{\text{RV} \\ \text{standard} \\ \text{normal}}} \le \underbrace{\frac{x - \mu}{\sigma}}_{\text{a number.}} \Big)$$

$$= P\left(Z \leq \frac{x-\mu}{\sigma} \right)$$

hence. $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.

To get the pdf, differentiate w.r.t. x.

$$f_X(x) = \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right).$$

differential of standard normal CDF is the standard normal pdf.

PDF of General Normal.

$$f_X(x) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

~~X_1, X_2 indept~~

$X_i \sim N(\mu_i, \sigma_i^2)$. independent

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

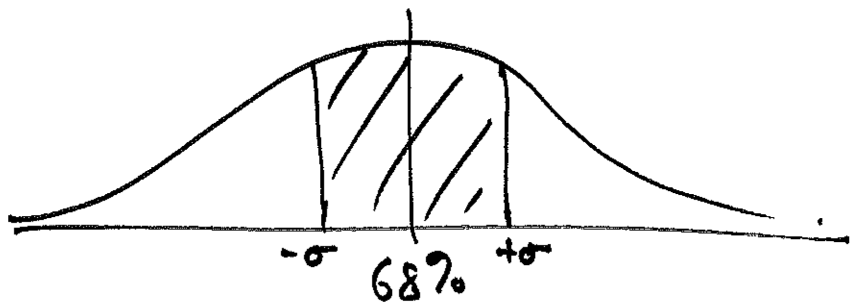
$$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2).$$

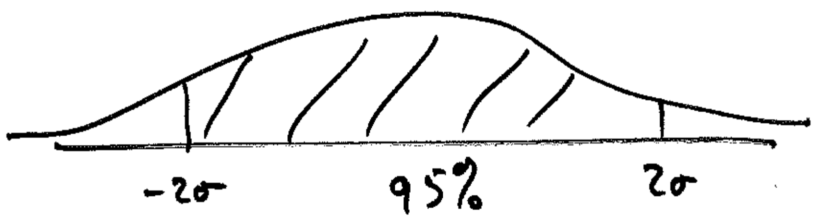
$X \sim N(\mu, \sigma^2)$

$P(|X-\mu| < \sigma) \simeq 0.68$

$P(|X-\mu| < 2\sigma) \simeq 95\%$

$P(|X-\mu| < 3\sigma) \simeq 0.997.$





$Y = \sum_{i=1}^{n} X_i \qquad Y \sim N(n\mu, n\sigma^2).$

If all $X_i \sim N(\mu, \sigma^2)$ and are independent

$M = \frac{Y}{n}$ - the mean of n iid $N(\mu, \sigma^2)$ RVs.

$M \sim N\left(\frac{n\mu}{n}, \frac{n\sigma^2}{n^2}\right).$

$M \sim N\left(\mu, \frac{\sigma^2}{n}\right).$

The distribution of the mean of n iid $N(\mu, \sigma^2)$ RV is also Normal, with the same expected value (μ), but with the variance reduced by a factor of $\frac{1}{n}$.

The variability in the mean of a set of Normal RVs reduces as we average over a larger and larger set.

$$E[X] = \sum x P(X=x) = \int x f_X(x)\, dx.$$

$$E[g(X)] \qquad Y = g(X)$$

$$P(Y=y) \quad \text{or} \quad f_Y(y)$$

$$E[Y] = \sum y P(Y=y) \quad \left(\text{or} \int y f_Y(y) dy\right)$$

$$E[g(X)] = \sum g(x) P(X=x).$$

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3 tosses of coin

Y = length of 1st run.

$$E[Y] = \sum_y y P(Y=y).$$

H H H	$\rightarrow$	3
H H T		2
H T H		1
H T T		1
T H H		1
T H T		1
T T H		2
T T T		3

$$P(Y=1) = \frac{1}{2}$$

$$P(Y=2) = \frac{1}{4}$$

$$P(Y=3) = \frac{1}{4}$$

$$E[Y] = \sum_{y=1,2,3} y P(Y=y) = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4}$$

$$= 1^{3}/_{4}$$

we can also find the expectation by summing over the individual elements of the state space.

$$\sum_{s} y \, P(\{s\})$$

$$3 \times \frac{1}{8} + 2 \times \frac{1}{8} + 1 \times \frac{1}{8} + 1 \times \frac{1}{8} + 1 \times \frac{1}{8}$$

$$+ 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} = 1\tfrac{3}{4}.$$

$E[g(X)]$

$$= \sum_{s \in S} g(X(s)) \, P(\{s\}).$$

sum over elements of the state space

$$= \sum_{x} \sum_{s : X(s) = x} g(X(s)) \, P(\{s\}).$$

sum over all elements, but divided up by $X(s)$.

$$= \sum_{x} \sum_{s : X(s) = x} g(x) \, P(\{s\}).$$

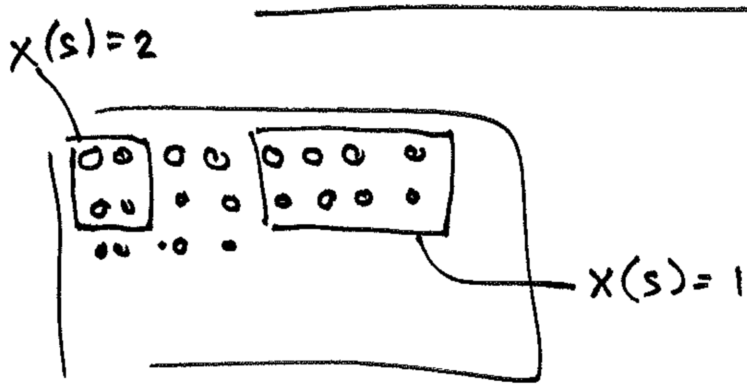
$X(s)$ maps the element of the state space to the value x.

$$= \sum_{x} g(x) \sum_{s : X(s) = x} P(\{s\}).$$

take $g(x)$ out of the inner summation as it doesn't depend on s hence

$$E[g(X)] = \sum_{x} g(x) \; P(X = x)$$

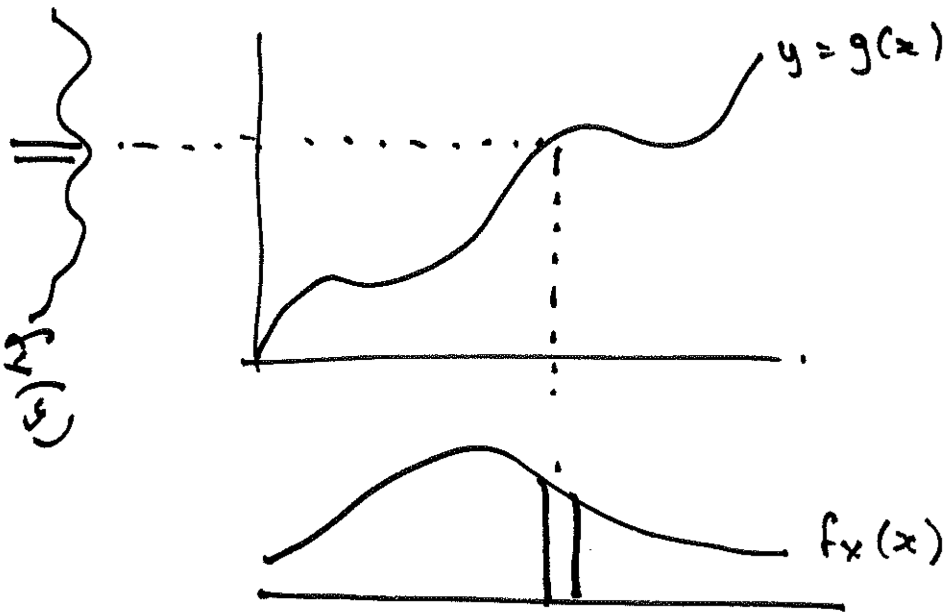
the second summation is over all elements of the state space where the RV maps the element to the value x.



Transformations

RV X $\quad$ $f_X(x)$

$Y = g(X)$ $\quad$ find $f_Y(y)$



$$f_X(x)\,\delta x = f_Y(y)\,\delta y$$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$\boxed{f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

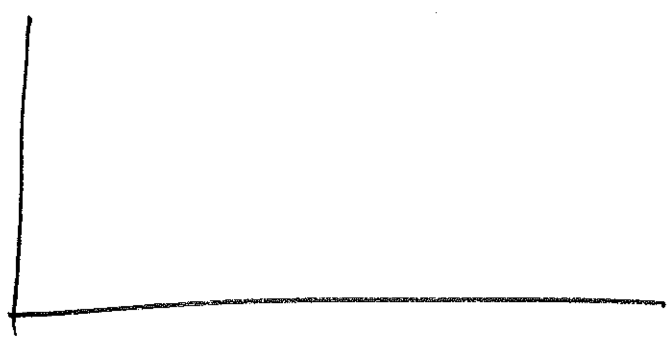
$$y = \mu + \sigma X \qquad x = \frac{y-\mu}{\sigma}$$

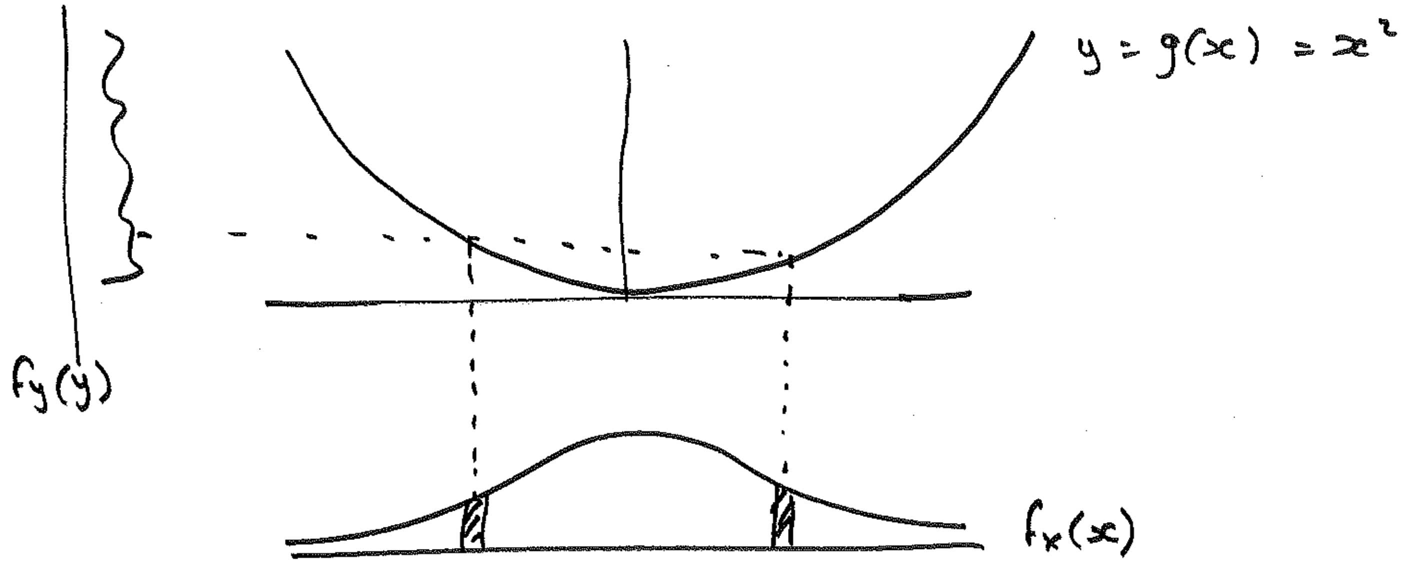
$$\frac{dy}{dx} = \sigma$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right). \times \frac{1}{\sigma}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right).$$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{\left|\frac{dy}{dx}\right|}.$$





Need to sum over the two values of x that map to the same value of y.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

$$y = x^2 \qquad \frac{dy}{dx} = 2x = 2\sqrt{y} \qquad g^{-1}(y) = \sqrt{y}$$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{\left|\frac{dy}{dx}\right|} \times 2$$

for the two x-values that map to the same y-value

$$= \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right) \times \frac{1}{2\sqrt{y}}$$

$$= \frac{2}{2\sqrt{y}\sqrt{2\pi}} \exp\left(\frac{-y}{2}\right) \qquad y \geq 0.$$

$$= \frac{1}{\sqrt{2\pi y}} \exp\left(\frac{-y}{2}\right).$$

pdf of RV that is the square of a N(0,1) RV.

$\underline{Y} = g(\underline{x})$ $\qquad$ $g: \mathbb{R}^n \to \mathbb{R}^n$

invertible and 1:1

joint pdf of Y.

$$f_Y(\underline{y}) = f_X(\underline{x}) \left| \frac{d\underline{x}}{d\underline{y}} \right|.$$

Jacobian of the transformation

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} & \cdots & \frac{dx_1}{dy_n} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} & \cdots & \frac{dx_2}{dy_n} \\ \vdots & & & \\ \frac{dx_n}{dy_1} & \cdots & \cdots & \frac{dx_n}{dy_n} \end{vmatrix}$$

J is determinant of the matrix