Binomial

$P(k \text{ successes in } n \text{ trials})$

$$p(k \text{ out of } n \mid \theta) = \binom{n}{k} \theta^k (1-\theta)^{n-k}.$$

We've run an experiment and observed 6 out of 10 successes.

What do we know about θ?

We represent what we know about θ using a pdf.

$$p(\theta \mid k \text{ out of } n) = \frac{p(k \text{ out of } n \mid \theta)\, p(\theta)}{p(k \text{ out of } n)}.$$

$$\propto \underbrace{\theta^k (1-\theta)^{n-k}}_{\text{this has the form of a Beta distribution}} \; * \; \underbrace{f_\Theta(\theta)}_{\text{if we chose this to also be a Beta distribution, then the product is also a Beta distribution.}}$$

LOTUS.

$$E(X) = \sum x P(X=x)$$

$$\int x f_X(x)\, dx$$

$Y = g(X)$ $\qquad E[Y] = \int y f_Y(y)\, dy$

we can derive $f_Y(y)$ from $f_X(x)$ and $g(x)$

OR $\qquad E(g(X)) = \int g(x) f_X(x)\, dx$

$$E(X^2) = \int x^2 f_X(x)\, dx.$$

Joint Distributions.

$$P(x_1, x_2) = P(x_1)P(x_2) \quad \text{if } X_1, X_2 \text{ independent.}$$

What happens if they are <u>not</u> independent?

How do we determine if they are independent?

$X_1, X_2 \sim$ Bernoulli, success probabilities may be different.

	$X_2 = 0$	$X_2 = 1$	
$X_1 = 0$	$2/6$	$1/6$	$\frac{3}{6}$
$X_1 = 1$	$2/6$	$1/6$	$\frac{3}{6}$
	$4/6$	$2/6$	

Joint PMF is specified by the 4 numbers inside the box. (≥ 0, sum to 1).

Marginal distribution for X_2

Marginal distribution for X_1

$P(x_1) =$

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2).$$

If X_1, X_2 are independent, the Joint PMF is the product of the marginal PMFs.

This is <u>true</u> in the example above.

		X_2		
		0	1	
X_1	0	$1/4$	0	$\frac{1}{4}$
	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
		$\frac{3}{4}$	$\frac{1}{4}$	

Marginal distribution $P(X_1)$

Marginal distribution $P(X_2)$

In this case X_1 and X_2 are <u>not</u> independent.

Joint CDF.

$$F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2)$$

Marginal CDFs

$$F_1(x_1) = P(X_1 \le x_1)$$

$$F_2(x_2) = P(X_2 \le x_2)$$

X_1 and X_2 are independent if.

$$F(x_1, x_2) = F_1(x_1) F_2(x_2)$$

joint pdf.

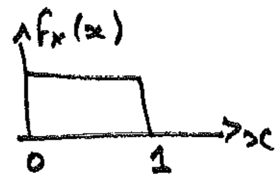
$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$$

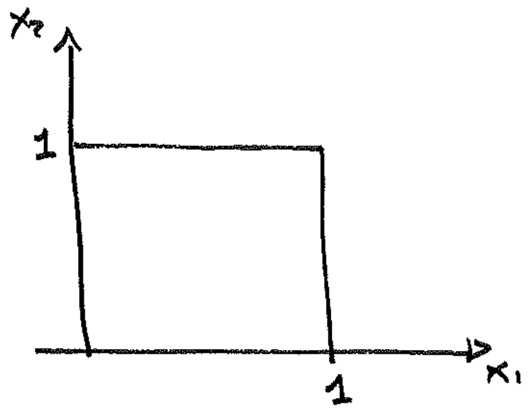
$$P(a \le x_1 \le b) = \int_a^b f_1(x_1)\, dx_1$$

$$P\big((x_1, x_2) \in B\big) = \iint_B f(x_1, x_2)\, dx_1\, dx_2$$

some specified region of the (x_1, x_2) plane

$X \sim U(0,1)$





$X_1 X_2$ takes values uniformly in the square.

Joint pdf is constant on the square, zero outside.

- Probability of (x_1, x_2) being in a specified region is proportional to the area of the region.

$$f(x_1, x_2) = \begin{cases} c & 0 \le x_1 \le 1, \; 0 \le x_2 \le 1) \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_0^1 \int_0^1 f(x_1, x_2)\, dx_1\, dx_2 = 1.$$

In this case $c = 1$

Marginal pdfs.

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

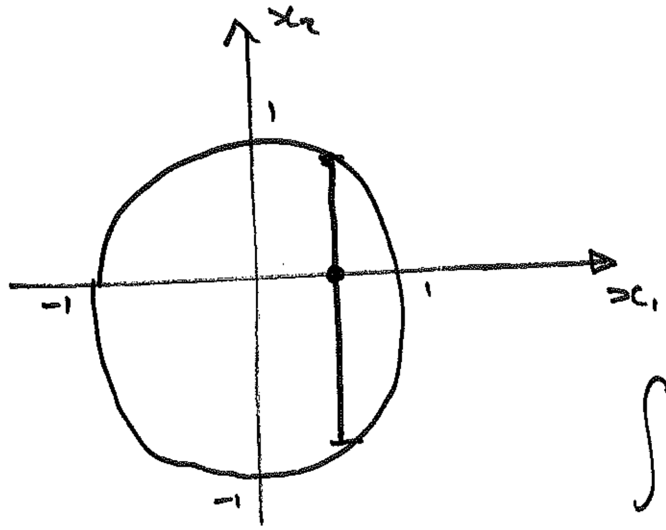
$$f_1(x_1) \sim U(0,1)$$

$$f_2(x_2) \sim U(0,1)$$

In this case

$$f(x_1, x_2) = f_1(x_1)\, f_2(x_2)$$ ← Joint is the product of the marginals.

and X_1 and X_2 are independent.



joint pdf of x_1, x_2 is uniform ~~in~~ inside the unit circle.

$$\iint_{\text{unit circle.}} c \, dx_1 \, dx_2 = 1$$

$$c \underbrace{\pi 1^2}_{\text{area of the circle}} = 1 \quad \Rightarrow \quad c = \frac{1}{\pi}$$

$$f(x_1; x_2) = \begin{cases} \frac{1}{\pi} & x_1^2 + x_2^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X_1, X_2 independent?

<u>Marginal distributions.</u>

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2$$

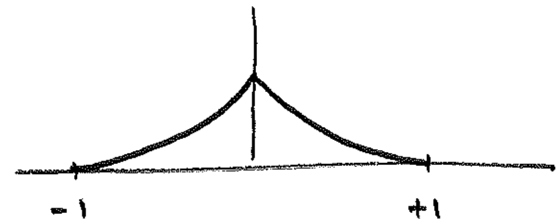
$f(x_1, x_2)$ is zero for $x_1^2 + x_2^2 > 1$

limits of integration are $x_2 = \pm\sqrt{1-x_1^2}$

$$f_1(x_1) = \int_{-\sqrt{1-x_1^2}}^{+\sqrt{1-x_1^2}} \frac{1}{\pi}\, dx_2$$

$$= \frac{2}{\pi}\sqrt{1-x_1^2}$$

$-1 \leq x_1 \leq 1$



$$f_2(x_2) = \frac{2}{\pi}\sqrt{1-x_2^2}$$

$f(x_1, x_2) \neq f_1(x_1) \times f_2(x_2)$ so X_1, X_2 are <u>not</u> independent.

They are not independent because of the constraint on x_1, x_2 that defines the region over which the pdf is non-zero.

conditional pdf.

$f_{X_2|X_1}(x_2|x_1)$ — if we (assume that) we know a value for x_1, what is the appropriate pdf for X_2?

$$f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

continuing the previous example.

$$f_{X_2|X_1}(x_2|x_1) = \frac{1/\pi}{\frac{2}{\pi}\sqrt{1-x_1^2}} = \frac{1}{2\sqrt{1-x_1^2}}$$

this is not a function of x_2

$\Rightarrow$ PDF of $f_{X_2|X_1}()$ is uniform.

$$f_{X_2|X_1}(x_2|x_1) \sim U\left(-\sqrt{1-x_1^2}, +\sqrt{1-x_1^2}\right)$$

$$f_{X_2|X_1}(x_2|x_1) \neq f_{X_2}(x_2)$$

— again, X_1, X_2 are not independent.

(if they were, the conditional dist. $f_{X_2|X_1}$ would not depend on X_1.)

$$E[g(X)] = \int g(x) f_X(x)\, dx$$

2D-LOTUS

$X_1, X_2 \sim f(x_1, x_2)$ joint pdf.

$g(x_1, x_2)$ - real valued fn of x_1, x_2

$$E[g(x_1, x_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$E[X_1 X_2] = E[X_1]E[X_2]$ if X_1, X_2 are independent.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \underbrace{x_1 f_{X_1}(x_1)}_{\text{constant wrt } x_2} \underbrace{\int_{-\infty}^{\infty} x_2 f_{X_2}(x_2)\, dx_2}_{E[X_2]}\, dx_1$$

$$= E[X_1]\, E[X_2]$$

TODAY'S QUIZ IS TAKE HOME.

- DO IT ON YOUR OWN
- TURN IT IN AT THE <u>START</u> OF CLASS ON THURSDAY.

Covariance + Correlation.

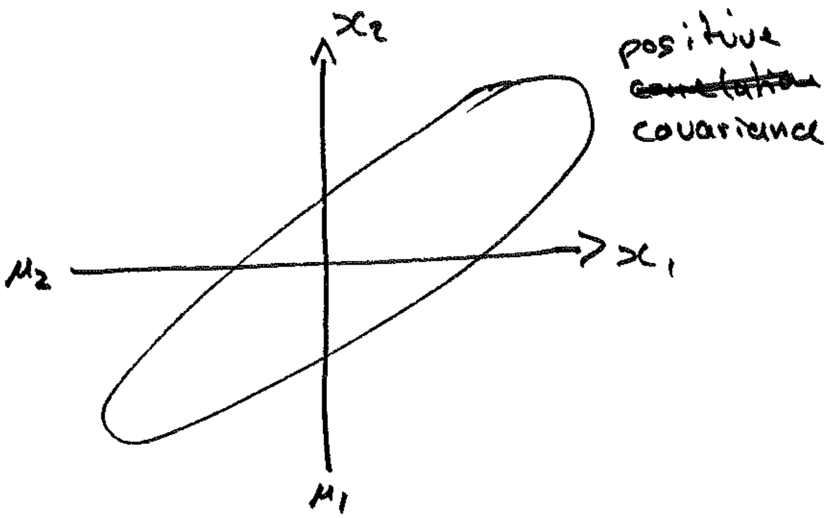
- Measure of how independent RVs are in a particular sense.
- will give us the variance of a sum of RV.

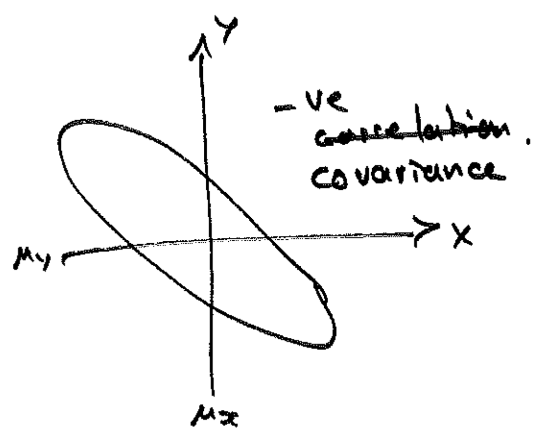
__Def.__

$$\text{Cov}(X, Y) = E\left[(X - E(X))(Y - E(Y)) \right]$$

if $x > \mu_x$ tends to imply $y > \mu_y$, then $\text{Cov}(X,Y) \geq 0$

if $x < \mu_x$ tends to imply $y < \mu_y$, then $\text{Cov}(X,Y) > 0$





Properties of covariance

(1) $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$

(2) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.

(3) $\mathrm{Cov}(X, c) = 0$ where c is a constant

(4) $\mathrm{Cov}(cX, Y) = c\,\mathrm{Cov}(X, Y)$

(5) $\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X, Y) + \mathrm{cov}(X, Z)$

Generalize.

$$\mathrm{Cov}\left(\sum_{i=1}^{m} a_i X_i \;,\; \sum_{j=1}^{n} b_j Y_j \right) = \sum_{i,j} a_i b_j \,\mathrm{cov}(X_i, Y_j)$$

Variance of the Sum of RVs.

$$\begin{aligned} \mathrm{Var}(X_1 + X_2) &= \mathrm{Cov}(X_1 + X_2 \,,\, X_1 + X_2) \\ &= \mathrm{Cov}(X_1, X_1) + \mathrm{Cov}(X_1, X_2) \\ &\quad + \mathrm{Cov}(X_2, X_1) + \mathrm{Cov}(X_2, X_2) \\ &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{Cov}(X_1, X_2) \end{aligned}$$

Hence $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$

iff $\mathrm{cov}(X_1, X_2) = 0$

when is $Cov(X_1, X_2) = 0$?

- if X_1, X_2 are independent.
- there are cases where dependent RVs have zero covariance.

ie if X_1, X_2 are independent, then $Cov(X_1, X_2) = 0$

but the converse does not necessarily hold.

ie $Cov(X_1, X_2) = 0$ does not guarantee independence.

$$Var(X_1 + X_2 + \dots X_n) = Var(X_1) + Var(X_2) + \dots + Var(X_n) + 2 \sum_{i<j} cov(X_i, X_j)$$