Some More Inequalities.

\[ \text{Cov}(X, Y) = \frac{E[XY]}{\sqrt{E[X^2]E[Y^2]}} \leq 1. \]

If \( X, Y \) are uncorrelated

\[ E[XY] = E[X]E[Y] \Rightarrow \text{the upper bound is likely to be very large.} \]

Consider zero mean \( X, Y \)

\[ \left| \text{Corr}(X, Y) \right| = \frac{E[XY]}{\sqrt{E[X^2]E[Y^2]}} \leq 1. \]

How tight the upper bound is depends on the correlation between \( X \) and \( Y \).
Jensen's Inequality

If \( g() \) is a convex function

then \( E[g(X)] \geq g(E[X]) \).

Because \( g(x) \) is convex, the tangent line is always below the curve.

\[ y = g(x) \]

\[ y = ax + b, \]

\[ g(\infty) \geq ax + b \]

\[ g(x) \geq ax + b \]

\[ E[g(x)] \geq E[ax + b] \]

\[ = aE[x] + b \]

\[ = a\mu + b \]

\[ = g(\mu) \]

\[ = g(E(x)) \]

hence \( E[g(X)] \geq g(E(X)) \).
Example.

Let $X$ be positive.

Let $g(x) = \frac{1}{x}$

$$E\left(\frac{1}{x}\right) \geq \frac{1}{E[x]}$$

$$E\left[\ln X\right] \leq \ln\left(E(X)\right)$$

$\ln$ is concave

so flip the inequality.

WLLN.

$$P\left(|\bar{X}_n - \mu| \geq c\right) \to 0 \quad \text{as} \quad n \to \infty$$

What about the distribution of $\bar{X}_n$?
Central Limit Theorem

\[ \frac{n^{-\frac{1}{2}}}{\sigma} (X_n - \mu) \to N(0,1) \text{ in distribution.} \]

\[ \text{as } n \to \infty \]

This is a Random Variable

\[ \text{as } n \to \infty \]

the RV\(X_n\)

has a Normal \((0,1)\) distribution

Equivalently

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i - n\mu \to N(0,1) \text{ in distribution} \]

\[ \text{as } \sum_{i=1}^{n} \text{ the sum of the } n \text{ iid RVs } X_i \]

has a normal distribution with

mean \(n\mu\)

std. dev. \(\sqrt{n}\sigma\)

\[ \overline{X}_n - \text{ sample mean, has a normal distribution with mean } \mu \]

\[ \text{std. dev. } \frac{\sigma}{\sqrt{n}} \]

Normal Approximation
As you average more and more i.i.d RVs the amount of variability gets smaller, and it gets smaller by a factor $\sqrt{n}$.

Proof: use MGF

Example.

Each minute a machine produces a length of rope with mean 240 ft, std. dev 5 inches.

Assuming that each minute's production is i.i.d., find (approximately) $P(\geq 250 \text{ ft in an hour})$.

Using the normal approximation

\[ \mu = 60 \times 4 = 240 \text{ ft} \]

\[ \sigma = \sqrt{60 \times 5} = 3.23 \text{ ft} \]

\[ P(X \geq 250) = 1 - \Phi \left( \frac{250 - 240}{3.23} \right) \]

\[ = 1 - \Phi(3.1) \approx 0.001 \]
**Continuity Correction.**

CLT still applies when Xi's are discrete.

E.g. \( X \) in # successes in 15 Bernoulli trials with \( p = 0.8 \)

\( X \sim \text{Binomial} \ (15, \ 0.3) \)

Approximate \( P(X = 4) \)

\[ E[X] = np = 15 \times 0.3 = 4.5 \]

\[ \text{Var}[X] = npq = 15 \times 0.3 \times 0.7 = 1.775^2 \]

\( \Rightarrow 1.775 \)

\[ P(X = 4) \approx P(3.5 \leq X \leq 4.5) \]

\[ = 0.5 - \Phi \left( \frac{-1}{1.775} \right) \]

\[ = 0.214 \]

Exact value from binomial distribution is 0.21866

Normal approximation is also commonly used to approximate probabilities from a Poisson distribution.
Binomial (n, p)

if \( p \) is very small

Skewed distribution.
Markov Chains.

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1. What proportion of the time will each of A, B, C have the ball?

2. Does this proportion depend on who had the ball to begin with? (more generally, PMF of the initial position?)