Some More Inequalities.

1) Cauchy - Schwartz

$$|E[XY]| \leq \sqrt{E(X^2)\,E(Y^2)}$$

if X, Y are uncorrelated

$$E[XY] = E[X]\,E[Y]$$ $\Rightarrow$ the upper bound is likely to be very large.

consider zero mean X, Y

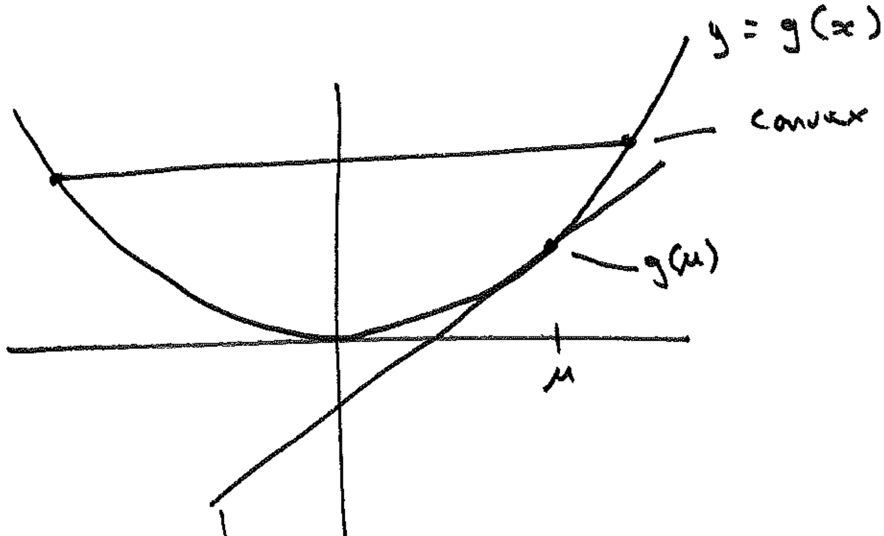
$$|\mathrm{corr}(X,Y)| = \frac{E[XY]}{\sqrt{E[X^2]\,E[Y^2]}} \leq 1.$$

How tight the upper bound is depends on the correlation between X and Y.

2, Jensen's Inequality

if $g()$ is a convex function

then $E[g(x)] \geq g(E[x])$.



convex - the curve is always below the line joining any two points on the curve.

tangent line at $(\mu, g(\mu))$

$y = ax + b$.

Because $g(x)$ is convex, the tangent line is always below the curve.

$$g(x) \geq ax + b$$

$$g(X) \geq aX + b$$

$$E[g(X)] \geq E[aX + b]$$

$$= aE[X] + b$$

$$= a\mu + b$$

$$= g(\mu)$$

$$= g(E(X))$$

hence

$$E[g(X)] \geq g(E(X))$$

Example.

let X be positive.

let $g(x) = \frac{1}{x}$

$$E\left(\frac{1}{x}\right) \geq \frac{1}{E[x]}$$

$$E[\ln X] \leq \ln(E(x))$$

ln fn is concave
so flip the inequality.

WLLN.

$$P(|\bar{X}_n - \mu| > c) \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

what about the distribution of $\bar{X}_n$?

Central Limit Theorem

$$\underbrace{\frac{n^{1/2}}{\sigma}(\bar{X}_n - \mu)}_{\text{this is a Random Variable}} \to \underbrace{N(0,1)}_{} \quad \text{in distribution.}$$

as $n \to \infty$

as $n \to \infty$
the RV
has a Normal $(0,1)$ distribution

Equivalently

$$\frac{\sum_{i=1}^{n} X_i - n\mu.}{\sqrt{n}\,\sigma} \to N(0,1) \quad \text{in distribution}$$

or: the sum of the n iid RVs X_i

has a normal distribution with

mean $n\mu$
std. dev. $\sqrt{n}\,\sigma$

or $\bar{X}_n$ - sample mean, has a normal distribution with mean μ
st. dev $\dfrac{\sigma}{\sqrt{n}}$

Normal Approximation

As you average more and more iid RVs the amount of variability gets smaller, and it gets smaller by a factor $\sqrt{n}$

Proof: uses MGF

Example.

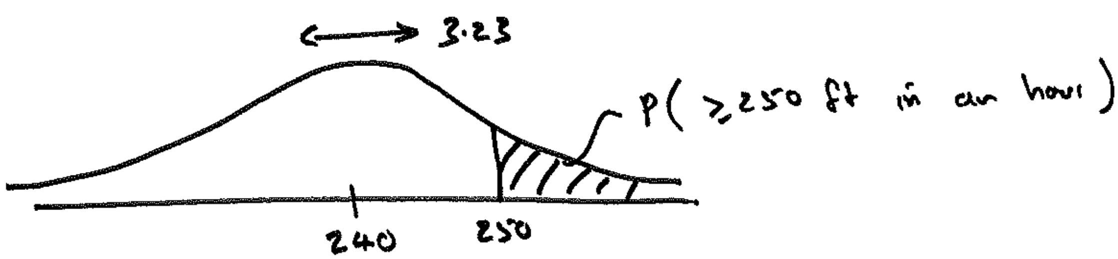
Each minute a machine produces a length of rope with mean 4ft, std-dev 5 inches.

Assuming that each minute's production is iid, find (approximately) $P(\geq 250 \text{ ft in an hour})$

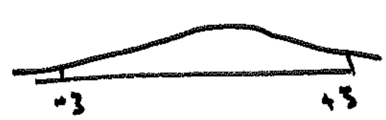
Using the normal approximation

$$\mu = 60 \times 4 = 240 \text{ ft}$$

$$\sigma = \sqrt{60} \times 5\text{in} = 3.23 \text{ ft}$$



$$P(X \geq 250) = 1 - \Phi\left(\frac{250-240}{3.23}\right)$$



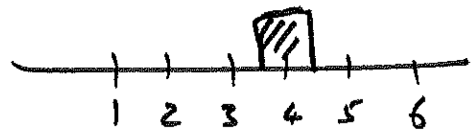
$$= 1 - \Phi(3.1) \simeq \underline{0.001}$$

Continuity Correction.

CLT still applies when X_i's are discrete.

Eg. X is # successes in 15 Bernoulli trials with $p = 0.3$

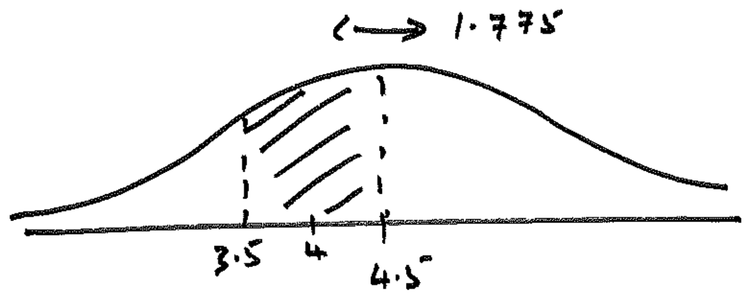
$X \sim \text{Binomial}(15, 0.3)$



Approximate $P(X = 4)$

$E[X] = np = 15 \times 0.3 = 4.5$

$\text{Var}[X] = npq = 15 \times 0.3 \times 0.7 = 1.775^2$



$$P(X=4) \simeq P(3.5 \leq X \leq 4.5)$$

$$= 0.5 - \Phi\left(\frac{-1}{1.775}\right)$$

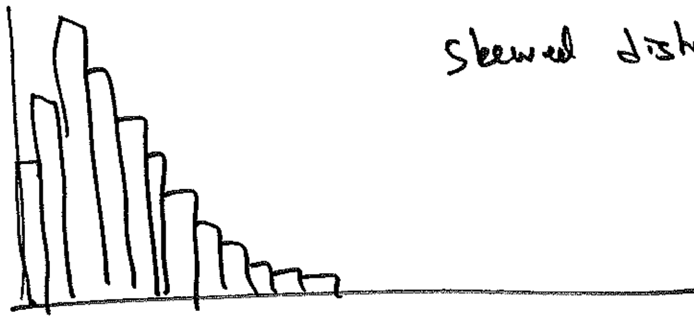
$$= 0.214$$

Exact value from binomial distribution is 0.21866

Normal approximation is also commonly used to approximate probabilities from a Poisson distribution.

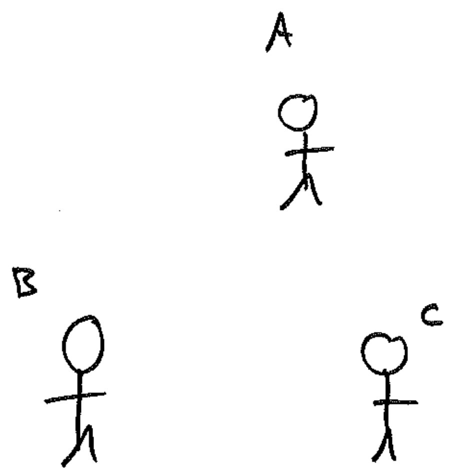
Binomial (n, p)

if p is very small



Skewed distribution.

Markov Chains.



From	to	Prob
A	B	0.2
	C	0.8
B	A	0.6
	C	0.4
C	A	~~0.5~~ 0.4
	B	~~[illegible]~~ 0.4
	C	0.2

1/ What proportion of the time will each of A, B, C have the ball?

2/ Does this proportion depend on who had the ball to begin with?

(More generally, PMF of the initial position?)