Probability.

frequency

classical - equally likely outcomes.

subjective

Experiments and Events.

set S - set of all outcomes of an experiment.

event is a subset of S

union / intersection / complement

$A \cup B$ $\quad$ $A \cap B$ $\quad$ A^c

counting. $P(A) = \dfrac{\#\text{ favoriable outcomes}}{\#\text{ outcomes}}$.

$= \dfrac{\#\text{ elements of } S \text{ that are in the event } A}{\#\text{ elements of } S}$.

$\underset{n_1}{\square} \quad \underset{n_2}{\square} \quad \underset{n_3}{\square} \longrightarrow n_1 \times n_2 \times n_3$ possibilities.

$\binom{n}{k}$ # ways of chosing k items out of n where order does not matter.

Sample	order matters	doesn't matter
with replacement	n^k	$\binom{n+k-1}{k}$
without replacement	$n(n-1)\dots(n-k+1)$	$\binom{n}{k}$

Non-nieve Definition

S - sample space

P - function that takes an event $A \subseteq S$ and returns $P(A) \in [0,1]$

$$P(\emptyset) = 0$$
$$P(S) = 1$$
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{if } A_i \text{ are disjoint}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

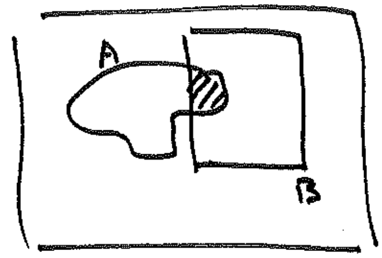
inclusion - exclusion

independence

$$P(A \cap B) = P(A)P(B) \quad \text{if } A, B \text{ independent.}$$

Conditional Probability

$$P(A|B) = \frac{P(A,B)}{P(B)}$$



$$P(A,B) = P(A|B)\,P(B)$$
$$P(B|A)\,P(A)$$

Independence : if $P(A|B)$ doesn't depend on B, then A, B are independent

$$\sum_b P(A,B) = P(A)$$ Marginalization

$$\sum_b P(A|B)\,P(B) = P(A)$$ LOTP

Bayes theorem.

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$

The denominator is often found using LOTP.

Random Variables.

Function from Sample Space $\rightarrow$ Real Line.

RV X

realization x

$X \sim$ Binomial (n, p) $\qquad \underbrace{P(X = k \mid n, p)}_{\text{PMF.}} = \binom{n}{k} p^k (1-p)^{n-k}$.

CDF event $X \leq x$

$$F(x) = P(X \leq x)$$

$$P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y) \quad \text{- independence.}$$

Expectation.

$$E[x] = \sum_x x P(X = x) \qquad \int x f_x(x) \, dx$$

$$E[X + Y] = E[X] + E[Y]$$

indicator RV.

$$X = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = P(A)$$

Some useful distributions

Bernoulli (p)

Binomial (n, p)

Hypergeometric

Geometric

Negative Binomial

Poisson

Continuous Distributions

PDF $f_X(x)$ if $P(a \le X \le b) = \int_a^b f_X(x)\, dx$.

$f_X(x) \ge 0$

$\int_{-\infty}^{\infty} f_X(x) = 1$

CDF $= P(X \le x) = \int_{-\infty}^{x} f_X(t)\, dt$

PDF $f_X(x) = \frac{d}{dx} F(x)$

Variance

$$Var(X) = E\left((X - E(X))^2 \right)$$

$$= E[X^2] - (E[X])^2 \geq 0$$

$U \sim Unif(0,1)$

$$F(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[g(X)] = \int g(x) f_X(x)\, dx$$

<u>or</u> transform $Y = g(X)$

find PDF of Y

Normal Distribution.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad N(0,1) \qquad \text{standard normal.}$$

transformation $Z = \dfrac{X-\mu}{\sigma}$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right)$$

Bivariate Normal.

covariance matrix $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$$f_X(\underline{x}) = \frac{1}{(2\pi)\,|\Sigma|^{1/2}} \exp\left(-\frac{1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu) \right).$$

Normal CDF $\Phi(z)$ of $N(0,1)$

$$\mathrm{Var}(X+c) = \mathrm{Var}(X)$$

$$\mathrm{Var}(cX) = c^2\,\mathrm{Var}(X).$$

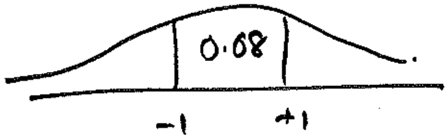
$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + \underbrace{2\,\mathrm{Cov}(X,Y)}_{=0 \text{ if } X,Y \text{ independent}}$$

$\mathrm{Cov}(X,Y)$ can $=0$ for some dependent RV X, Y

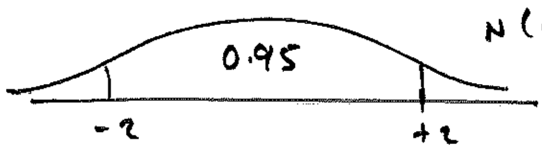
Sums of ~~Normal~~ RVs.

$$Y = \frac{1}{N} \sum_{i=1}^{N} X_i \qquad X_i \sim \text{iid, mean } \mu, \text{ variance } \sigma^2$$

$$Y \sim N\left(\mu, \frac{\sigma^2}{N}\right)$$

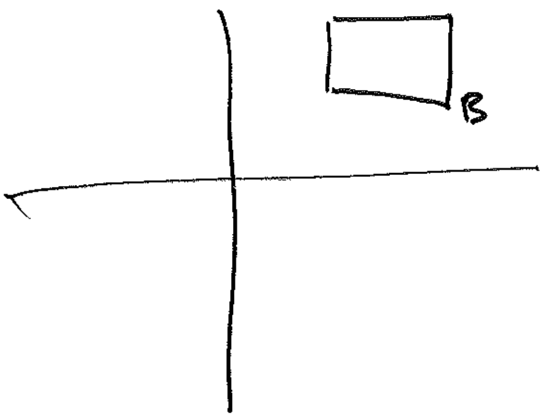


$N(0,1)$



$N(0,1)$

$f_{X,Y}(x,y)$



$P(X,Y \in B)$

$$= \iint_B f_{X,Y}(x,y)\, dx\, dy$$

Marginalization

$$f_X(x) = \int_y f_{X,Y}(x,y)\, dy$$

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$$

<u>Covariance + correlation.</u>

$$\text{Cov}(X,Y) = E((X-E(X))(Y-E(Y)))$$

$$= E[XY] - E[X]E[Y]$$

$$\text{Var}(X+Y) = \text{Cov}(X+Y,\ X+Y)$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y)$$

$\text{Corr}(X,Y) \equiv$ Cov of standardized versions of X, Y

$$\text{Cov}\left(\frac{X-\mu_X}{\sigma_X},\ \frac{Y-\mu_Y}{\sigma_Y}\right)$$

$$-1 \leq \rho \leq 1$$

Transformations

$$Y = g(X)$$

$$f_Y(y) = f_X(x)\,\frac{dx}{dy}$$

Inequalities.

Markov Chains

$$T = \begin{array}{cc} & \text{To} \\ \text{from} & \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \end{array}$$

multi-step transition probabilities T^n

$$\underline{s}_n = \underline{s}_0 T^n$$

for irreducible, aperiodic Markov chain

$\underline{s}_n \to \underline{v}$ as $n \to \infty$, irrespective of $\underline{s}_0$

└ stationary distribution

$$\underline{v} = \underline{v}\,T$$

and sum of elements of v is 1.

Have N items, labelled $1 \dots n$

Sample k without replacement.

what's the probability that the sample maximum is m?

$$P(M = m \mid K = k, N = n)$$

S has $\binom{n}{k}$ elements.

How many of these are of interest?

For the largest number to be m, the remaining items ($k-1$ of them), must be chosen from those labelled $1 \dots (m-1)$

$\Rightarrow$ there are $\binom{m-1}{k-1}$ element of interest in S

$$\Rightarrow P(M = m \mid K = k, N = n) = \frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

What is $P(N = n \mid K = k, M = m)$?

$$P(N=n \mid M=m,\ K=k)$$

$$= \cancel{P(M=m \mid K=k, N=n}$$

$$= \frac{P(M=m \mid N=n,\ K=k)\ P(N=n \mid K=k)}{P(M=m \mid K=k)}$$

$P(N=n \mid K=k)$ - captured k deer, without looking at any of the tags, what do we think the PMF for N is?

$$= U(k, N_{max})$$

$$P(M=m \mid K=k) = \cancel{\sum_{n} p(M=m, K=k, N=n}$$

$$= \sum_{n} p(M=m, N=n \mid K=k)$$

$$= \sum_{n} p(M=m \mid N=n, K=k)\ P(N=n \mid K=k)$$

$$\sum_{n=0}^{\infty} \approx \frac{\binom{m-1}{k-1}}{\binom{n}{k}} \quad \frac{1}{N_{max} - k}$$

$$P(N = n \mid M = m, K = k)$$

$$= \frac{\dfrac{\binom{m-1}{k-1}}{\binom{n}{k}} \quad \dfrac{1}{N_{max} - k}}{\dfrac{1}{N_{max}-k} \displaystyle\sum_{n=0}^{N} \dfrac{\binom{m-1}{k-1}}{\binom{n}{k}}}$$

$$= \begin{cases} \dfrac{\binom{m-1}{k-1} / \binom{n}{k}}{\displaystyle\sum_{n=\cancel{k}m}^{N_{max}-1} \binom{m-1}{k-1} / \binom{n}{k}} & m < n < N_{max} \\ 0 & \end{cases}$$

what is $\displaystyle\sum_{n=k}^{N_{max}-1} \binom{m-1}{k-1} / \binom{n}{k}$

$$= \binom{m-1}{k-1} \sum_{n=\cancel{k}m}^{N_{max}-1} \frac{1}{\binom{n}{k}}$$

$$\sum_{n=m}^{\infty} \frac{1}{\binom{n}{k}} = \frac{k}{k-1} \frac{1}{\binom{m-1}{k-1}}$$ ToF.

$$P(N=n \mid M=m, K=k) = \frac{k-1}{k} \frac{\binom{m-1}{k-1}}{\binom{n}{k}}.$$

$$E[N] = \sum n P(N=n \mid M=m, K=k)$$

$$= \frac{(k-1)(m-1)}{k-2}$$

collect $k=10$ deer, highest tag is $m=40$

$$E[N] = \frac{9 \times 39}{8} = 43.875$$

$k=4$ $m=20$

$$E[N] = \frac{3 \times 19}{2} = 28.5$$

Conditional PMF / distribution

Bayes thm. - reverse the conditioning.

LOTP / marginalization to obtain the denominator

Expected values.