AMS131 - Quiz 3
Thursday 31th May, 2018.
You must show working/explain all answers for full credit.

Some (possibly) useful results:
\[
\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] \\
\text{Var}(aX) = a^2\text{Var}(X) \\
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)
\]

1. Suppose that \(X, Y,\) and \(Z\) are three random variables such that \(\text{Var}(X) = 1, \text{Var}(Y) = 4,\)
\(\text{Var}(Z) = 8, \text{Cov}(X, Y) = 1, \text{Cov}(X, Z) = -1,\) and \(\text{Cov}(Y, Z) = 2.\) What is \(\text{Var}(3X - Y - 2Z + 1)\)?

\[
\text{Var}(3X - Y - 2Z + 1) = 9\text{Var}(X) + \text{Var}(Y) + 4\text{Var}(Z) \\
+ 2\text{Cov}(3X, -Y) + 2\text{Cov}(3X, -2Z) \\
+ 2\text{Cov}(-Y, -2Z) \\
= 9\text{Var}(X) + \text{Var}(Y) + 4\text{Var}(Z) \\
- 6\text{Cov}(X, Y) - 12\text{Cov}(X, Z) \\
+ 4\text{Cov}(Y, Z) \\
= 9 + 4 + 32 \\
- 6 + 12 + 8 \\
= 59
\]
2. A fair die is rolled, and then a coin with probability $p$ of Heads is flipped as many times as the die roll says, e.g., if the result of the die roll is a 3, then the coin is flipped 3 times. Let $X$ be the result of the die roll and $Y$ be the number of times the coin lands Heads. Find the joint PMF of $X$ and $Y$. Are they independent?

\[
P(X, Y) = P(Y|X) P(X).
\]

\[
P(Y|X) = \text{prob } y \text{ successes in } X \text{ times} = \binom{x}{y} p^y (1-p)^{x-y}
\]

\[
P(X) = \frac{1}{6}
\]

\[
P(X, Y) = \frac{1}{6} \binom{x}{y} p^y (1-p)^{x-y} \quad \text{for } x = 1, 2, \ldots, 6 \quad y = 0, 1, \ldots, x.
\]

They are not independent.

\[
P(Y|X) \text{ depends on } x
\]

3. Random variable $X$ has the uniform distribution on the interval $[-2, 2]$ and $Y = X^6$. Show that $X$ and $Y$ are uncorrelated.

\[
\text{cov}(X, Y) = E[XY] - E[X]E[Y]
\]

\[
E[X] = 0 \text{ by symmetry.}
\]

\[
E[XY] = E[X^7]
\]

$a^7$ is an odd function.

\[
\Rightarrow E[X^7] = 0 \text{ as the range is symmetric.}
\]

Hence

\[
\text{cov}(X, Y) = 0 - 0 E[Y] = 0
\]

and $P(X, Y) = 0$ i.e. uncorrelated.