Sample Exam Questions

1) $E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$

$= E[X^2] + 2E[XY] + E[Y^2]$

X, Y are iid so $E[Y^2] = E[X^2]$

and $E[XY] = E[X]E[Y]$

and $E[Y] = E[X]$

$= 2E[X^2] + 2(E(E))^2$

2) $X = X_1 + X_2 + \dots + X_r$

X_1 : # failures before 1st success.

X_2 : # failures befor 2nd success

$\vdots$

X_r # failures before r^{th} success.

The X_i's are iid Geometric (p)

$Var(X) = Var(X_1 + X_2 + \dots + X_r)$

$= Var(X_1) + Var(X_2) + \dots + Var(X_r)$

Variance of a Geometric (p) is $\dfrac{1-p}{p^2}$

$\Rightarrow$ Variance of Negative Binomial (r, p) is $\dfrac{r(1-p)}{p^2}$

3, $X, Y \sim$ iid $N(0,1)$.

$W = X^2 + Y^2$

$f_W(w) = \frac{1}{2} e^{-w/2}$

a) $R^2 = W$

$R = W^{1/2}$

$f_R(r)\, dr = f_W(w)\, dw$

$f_R(r) = f_W(w) \frac{dw}{dr}$ $\qquad \frac{dw}{dr} = 2R = 2\sqrt{w}$

$= \frac{1}{2} e^{-R^2/2} \cdot 2R$

$= r e^{-r^2/2}$

b) $P(X > 2Y + 3)$

$= P(X - 2Y > 3)$

RV $X - 2Y$ is Normal, mean 0

variance 5

Hence $P(X - 2Y > 3)$ is same as $N(0,5) > 3$

$= 1 - \Phi\left(\frac{3}{\sqrt{5}}\right)$

4.

a) $E(U_1) = \frac{1}{2}$.

$\text{Var}(U_1) = \frac{1}{12}$.

Mean of X : $E(X) = E(U_1) + E(U_2) + \dots E(U_{60})$ linearity of expectation

$= 60\, E(U_1)$ symmetry

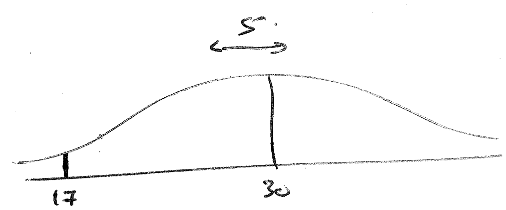
$= 30$

$\text{Var}(X) = \text{Var}(U_1) + \text{Var}(U_2) + \dots \text{Var}(U_{60})$ as U's are independent.

$= \frac{60}{12} = 5$

b). $P(X > 17)$.

approximate pdf of x by $N(30, 5)$.



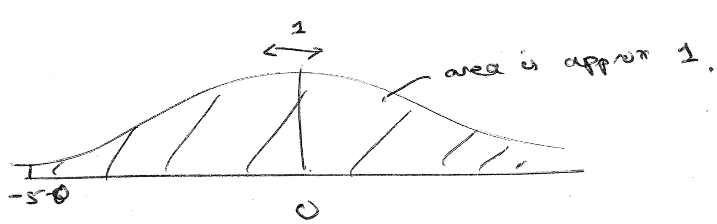
$P(X > 17)$

$\cancel{\simeq P(Z > 17k}$

$\simeq P\left(\frac{X - 30}{\sqrt{5}} > \frac{17 - 30}{\sqrt{5}} \right)$.

$\simeq P\left(Z > \cancel{4} - \frac{13}{\sqrt{5}}\right)$

$-\frac{13}{\sqrt{5}} \simeq -5.8$.



$P(X > 17) \simeq 1$.

5/ Number the 48 non-aces 1 through 48.

Let X_i be indicator that card with number i is dealt before any of the aces.

$X = X_1 + X_2 + \ldots + X_{48}$ is total # of cards before 1st ace.

and, ~~b~~ using linearity of expectation and symmetry.

$E[X] = 48 \, E[X_1]$.

Expectation of an indicator RV is the prob. of the indicator RV taking the value 1.

What's the prob of card 1 being dealt before any of the 4 aces?

Consider the orderings of the 5 cards. One of ~~them~~ the 5 possible orderings results in card 1 being dealt before any of the 4 aces.

$\Rightarrow \; E[X_1] = \frac{1}{5}$

$\Rightarrow \; E[X] = \frac{48}{5}$

6/

$$P(1^{st} \text{ six appears on roll } k) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right) \qquad k = 1, 2, \ldots$$

Event that Bob wins is all odd k.

These are disjoint, so $P(\text{Bob wins}) = \sum_{k \text{ odd}} P(1^{st} \text{ six appears on roll } k)$.

k odd so k-1 even.

$$= \sum_{j=1}^{\infty} \left(\frac{5}{6}\right)^{2j-2}\left(\frac{1}{6}\right)$$

$$= \frac{1}{6}\sum_{j=1}^{\infty}\left(\frac{5}{6}\right)^{2j-2}$$

$$= \frac{1}{6}\sum_{j=1}^{\infty}\left(\frac{25}{36}\right)^{j-1}$$

$$= \frac{1}{6}\sum_{j=0}^{\infty}\left(\frac{25}{36}\right)^{j}$$

$$= \frac{1}{6}\,\frac{1}{1-\frac{25}{36}}$$

$$= \frac{6}{11}$$

7,

$$\frac{\binom{5}{3}\binom{5}{3}}{\binom{10}{6}}$$

8, a)

P (dry on at least one of next 3 days)

= 1 − P(rain on all 3 of next 3 days).

= 1 − P (wet tomorrow | wet today) × P(wet in 2 days time | wet tomorrow)

× P(wet in 3 days time | wet tomorrow)

$= 1 - 0.4 \times 0.4 \times 0.4$

$= 0.936.$

b) PDF for friday is given by

$S T^2$

$$ST = [0.2 \quad 0.8] \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = [0.62 \quad 0.38].$$

$$ST^2 = [0.62 \quad 0.38] \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = [0.662 \quad 0.338]$$

$\Rightarrow$ Prob. (wet on friday) $= \underline{0.338}.$